

# Notes on 4d Yang-Mills Theory

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In these notes I'll introduce you to Yang-Mills theory on the Riemannian 4-manifold  $\mathbb{R}^4$ , from the point of view taken by [Cos11] and [EWY18] (much of the text in this note comes from the latter).

## 1 Yang-Mills in the Classical BV Formalism

Let  $G$  be a compact simple <sup>1</sup> Lie group, and let  $V$  be a finite-dimensional representation of  $G$  equipped with a non-degenerate invariant pairing  $\mu: V \otimes V \rightarrow \mathbb{R}$ . The usual description of Yang-Mills theory, in what's known as the second-order formalism, is given as follows.

The *fields* of Yang-Mills theory are a gauge field  $A \in \Omega^1(\mathbb{R}^4; \mathfrak{g})$  and a spinor  $\psi \in \Omega^0(\mathbb{R}^4; S \otimes V)$ , where  $S \cong S_+ \oplus S_-$  is the Dirac spinor bundle. The (infinitesimal) gauge transformations are controlled by the Lie algebra  $\Omega^0(\mathbb{R}^4; \mathfrak{g})$ , with a gauge transformation  $c$  acting on the fields by

$$\begin{aligned} A &\mapsto dc + [c, A] \\ \psi &\mapsto \alpha(c)(\psi) \end{aligned}$$

where  $\alpha$  is the derivative of the representation  $G \rightarrow \text{Aut}(V)$ .

In order to define the *action* of Yang-Mills theory we choose a non-degenerate  $G$ -invariant pairing  $\mu: V \otimes V \rightarrow \mathbb{R}$ , and a positive operator  $m: V \rightarrow V$  – the *mass matrix* of the fermions. We will also write  $\rho$  for the Clifford multiplication map  $\Omega^1(\mathbb{R}^4; S) \rightarrow \Omega^0(\mathbb{R}^4; S)$ . The Yang-Mills action is the functional

$$S(A, \psi) = \int_{\mathbb{R}^4} \frac{1}{2} \|F_A\|^2 + \mu(\psi, (\not{d}_A + m)\psi),$$

where  $F_A = dA + g[A, A]$  <sup>2</sup>, and  $\not{d}_A \psi = \rho(d\psi + g\alpha(A)\psi)$ . The norm of  $F_A$  is computed using the standard metric on  $\mathbb{R}^4$  together with a non-degenerate invariant pairing on the Lie algebra  $\mathfrak{g}$ .

We can fit Yang-Mills theory into the framework we've heard about in Phil's lecture by computing the classical BV complex. As a cochain complex, the classical BV complex takes the form

$$\begin{array}{ccccccc} & & \underline{0} & & \underline{1} & & \underline{2} & & \underline{3} \\ & & & & & & & & \\ \text{Fermion degree 0} & & \Omega^0(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d} & \Omega^1(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d^*d} & \Omega^3(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d} & \Omega^4(\mathbb{R}^4; \mathfrak{g}) \\ & & & & & & & & \\ \text{Fermion degree 1} & & & & \Omega^0(\mathbb{R}^4; S \otimes V) & \xrightarrow{m+\not{d}} & \Omega^0(\mathbb{R}^4; S \otimes V) & & \end{array}$$

<sup>1</sup>We could consider semisimple too – we're just taking simple to make things easier, there will only be one coupling constant.

<sup>2</sup> $g$  here is a real number called the *coupling constant*.

placed in cohomological degrees 0, 1, 2, 3. The order 3 and 4 parts of the BV action functional define a classical interaction in  $\mathcal{O}_{\text{loc}}(\mathbb{R}^4)$  that solves the classical master equation: it looks, concretely, like

$$I = g \text{Tr}(F_A \wedge *[A, A]) + \mu(\psi, (\not{A})\psi) + ([c, A], A^\vee) + ([c, \psi], \psi^\vee) + ([c, c], c^\vee).$$

This theory has a problem: it doesn't admit a gauge-fixing operator, so we can't perform BV quantization. We get around this problem by passing to a homotopy equivalent theory called the *first order formalism* for Yang-Mills.

## 1.1 The First Order Formalism

The problem with the classical BV complex above appears in the second order differential  $d * d$ . The idea of the second order formalism is that we introduce a Lagrange multiplier field  $B$  – a self-dual  $\mathfrak{g}$ -valued 2-form – along with an equation of motion saying that  $B = F_A$ , and then replace the term  $\|F_A\|^2$  in the action by a term like  $\text{Tr}(B \wedge *F_A)$ . So, the new action functional looks like

$$S_{\text{FO}}(A, B, \psi) = \int_{\mathbb{R}^4} \langle F_A, B \rangle - \frac{1}{2} \|B\|^2 + \mu(\psi, (\not{A} + m)\psi).$$

The classical BV complex in first-order Yang-Mills theory is, as a cochain complex,

$$\begin{array}{ccccccc} & \underline{0} & & \underline{1} & & \underline{2} & & \underline{3} \\ \text{Fermion degree 0} & \Omega^0(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d} & \Omega^1(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d_+} & \Omega^2_+(\mathbb{R}^4; \mathfrak{g}) & & \\ & & & & \nearrow^{-\text{id}} & & & \\ \text{Fermion degree 0} & & & \Omega^2_+(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d} & \Omega^3(\mathbb{R}^4; \mathfrak{g}) & \xrightarrow{d} & \Omega^4(\mathbb{R}^4; \mathfrak{g}) \\ \text{Fermion degree 1} & & & \Omega^0(\mathbb{R}^4; S \otimes V) & \xrightarrow{m+\not{d}} & \Omega^0(\mathbb{R}^4; S \otimes V) & & \end{array}$$

It comes equipped with a classical local interaction functional which takes the form

$$I = g \text{Tr}(B \wedge *[A, A]) + \mu(\psi, (\not{A})\psi) + ([c, A], A^\vee) + ([c, \psi], \psi^\vee) + ([c, c], c^\vee).$$

This theory is homotopy equivalent to second-order Yang-Mills theory, as defined above, plus an auxiliary form  $B$  with trivial action  $\langle B, B \rangle$ . The idea of the equivalence is that we can do an upper triangular change of variables, sending  $B$  to  $B + (F_A)_+$ : we can turn this into a path in the space of action functionals.

## 2 How Quantization Works

Now, let's start to talk about the construction of the effective action functional  $I[L]$  at scale  $L$ , and its behaviour under local RG flow. In particular, we can start to explain how to compute the 1-loop  $\beta$ -function for Yang-Mills theory.

Let me remind you of the following story: the steps for quantizing a theory in the BV formalism.

1. First, choose a gauge-fixing operator  $Q^{\text{GF}}$  for the BV differential  $Q$ . The commutator  $D = [Q, Q^{\text{GF}}]$  defines a generalized Laplacian.

2. Calculate the heat kernel  $K_t$  mollifying the integral kernel for the operator  $D$ . Usually, we think of the complex  $\mathcal{E}$  as being divided into summands corresponding to BV fields in the theory. The heat kernel  $K_t$  will likewise split as a sum over pairs of fields  $\phi, \phi^\vee$  paired together under the symplectic structure.
3. Use  $Q^{\text{GF}}$  to calculate the propagator  $P(\varepsilon, L)$  for the theory. Just like the heat kernel, the propagator splits into a sum over pairs of BV fields.
4. Now the key step, the introduction of counter-terms. We've learned that given a pretheory  $\{I[L]\}$  it's possible to obtain a local functional  $I$  in the limit of homotopy RG flow as  $\varepsilon \rightarrow 0$ . Counter-terms allow us to go the other way, i.e. describe a family of interactions at scale  $L$  starting from a local functional  $I$ . The idea is to find some local functionals  $I^{\text{CT}}(\varepsilon)$  depending smoothly on  $\varepsilon$ , so that the limit

$$I[L] = \lim_{\varepsilon \rightarrow 0} W(P(\varepsilon, L), I - I^{\text{CT}}(\varepsilon))$$

exists. The counterterms  $I^{\text{CT}}(\varepsilon)$  will be singular in the small  $\varepsilon$  limit, but it's always possible to find such functionals in the space  $\mathcal{O}_{\text{loc}}(\mathbb{R}^4)[[\hbar]][\varepsilon^{\pm 1}, \log \varepsilon]$ . This procedure is very non-canonical, one needs to choose something called a *renormalization scheme* in order to fix them: roughly a splitting of the space of  $\varepsilon$ -dependent local functionals into functionals singular at  $\varepsilon$ , and regular at  $\varepsilon$ .

5. Call the tentative interaction  $\tilde{I}[L]$ . The result is a pre-theory, not necessarily a theory yet: the QME might fail to hold. We can try to modify the interaction even more, by adding order-by-order in  $\hbar$  a further correction  $J$  cancelling the failure to solve the QME. This is *not* always possible, there is a cohomological obstruction. At level  $n + 1$  we need to find a potential  $J$  in  $\mathcal{O}_{\text{loc}}$  for the functional

$$O_{n+1} = \hbar^{-n-1}(Q\tilde{I}[L]) + \{\tilde{I}[L], \tilde{I}[L]\}_L + \Delta_L \tilde{I}[L].$$

Now, with this story in mind, we can calculate the 1-loop  $\beta$ -function using the following result.

**Definition 2.1.** The *1-loop  $\beta$ -function*  $\beta^{(1)}$  is the cohomology class of the logarithmic derivative of the 1 loop term in the family  $R_\lambda I[L]$  under local RG flow.

Remember from Kevin's lecture that, under local RG flow, to first order, the coupling varies by a term proportional to  $\log \lambda$ . We're really interested in the sign: the coupling will end up becoming small at large  $\lambda$  as long as the sign is *negative*.

**Theorem 2.2.** In a nice enough quantum field theory (classically translation and scale-invariant, and marginal at 1-loop),  $\beta^{(1)}$  can be identified with the cohomology class in the classical complex  $(\mathcal{O}_{\text{loc}}(\mathbb{R}^4), Q + \{I, -\})$ , of the logarithmic 1-loop counter-term  $I_{\log}^{\text{CT}}$ .

So how do we compute  $I_{\log}^{\text{CT}}$ ? Well, as we described above, we're supposed to look at the  $\varepsilon \rightarrow 0$  limit of the part of the weight  $W(P(\varepsilon, L), I)$  associated to the 1-loop Feynman diagrams, and extract the term asymptotic to  $\log(\varepsilon)$  – this is the part that the logarithmic counter-term is designed to cancel. We then determine its cohomology class, which is well-defined (i.e. independent of the choice of renormalization scheme). The result is the following, first demonstrated in a very different framework by Gross, Wilczek and Politzer (in 1973).

**Theorem 2.3.** The one-loop  $\beta$ -function of Yang–Mills theory is equal to

$$\beta^{(1)}(g) = -\frac{g^3}{16\pi^2} \left( \frac{11}{3}C(\mathfrak{g}) - \frac{4}{3}C(V) \right)$$

where  $C(\mathfrak{g}) \text{ id}_{\mathfrak{g}}$  and  $C(V) \text{ id}_V$  are the quadratic Casimir invariants for the representations  $\mathfrak{g}$  and  $V$  of  $G$  respectively.

**Corollary 2.4.** Yang–Mills with gauge group  $\text{SU}(n)$  and with  $f$  fundamental flavors is asymptotically free (has negative 1-loop  $\beta$ -function) if  $f < (11/2)n$ .

### 3 Quantization of Yang-Mills Theory

Ok, let's try to quantize this classical field theory to a quantum field theory, according to Phil's definition. So first we need to understand the scale  $L$  BV operator, and then we need to cook up effective interactions  $I[L]$  at level  $L$ . There's a procedure for doing this by introducing *counter-terms* to modify the interaction – we can calculate these at least to first order.

#### 3.1 Gauge Fixing

As a first step, we have to define a gauge-fixing operator  $Q^{\text{GF}}$  for first-order Yang-Mills theory. There's a fairly obvious guess, using the adjoint operator  $d^*$  to the de Rham differential.

$$\begin{aligned} \Omega^0(\mathbb{R}^4; \mathfrak{g}) &\xleftarrow{d^*} \Omega^1(\mathbb{R}^4; \mathfrak{g}) \xleftarrow{2d^*} \Omega_+^2(\mathbb{R}^4; \mathfrak{g}) \\ \Omega_+^2(\mathbb{R}^4; \mathfrak{g}) &\xleftarrow{2d_+^*} \Omega^3(\mathbb{R}^4; \mathfrak{g}) \xleftarrow{d^*} \Omega^4(\mathbb{R}^4; \mathfrak{g}) \\ \Omega^0(\mathbb{R}^4; S \otimes V) &\xleftarrow{d-m} \Omega^0(\mathbb{R}^4; S \otimes V). \end{aligned}$$

In order to show that this defines a gauge fixing operator we must compute the operator  $[Q, Q^{\text{GF}}]$ , and check that it's a generalized Laplacian. In the pure gauge sector, it's the sum of two terms: the usual Laplacian on differential forms, plus a first-order operator  $D_{\text{vert}}$  defined by

$$\begin{array}{ccc} \Omega^0(\mathbb{R}^4; \mathfrak{g}) & \Omega^1(\mathbb{R}^4; \mathfrak{g}) & \Omega_+^2(\mathbb{R}^4; \mathfrak{g}) \\ & \uparrow -2d^* & \uparrow -2d_+^* \\ \Omega_+^2(\mathbb{R}^4; \mathfrak{g}) & \Omega^3(\mathbb{R}^4; \mathfrak{g}) & \Omega^4(\mathbb{R}^4; \mathfrak{g}). \end{array}$$

Restricted to the fermions, the operator  $D = [Q, Q^{\text{GF}}]$  is clearly just the usual Laplacian – obtained as the square of the Dirac operator – minus the identity times  $m^2$ . Therefore the total generalized Laplacian is the sum of two terms:

$$D = (\Delta_\Omega - m^2 \text{id}_{\text{matter}}) + D_{\text{vert}}$$

where  $\Delta_\Omega$  is the usual Laplacian operator on differential forms, and  $D_{\text{vert}}$  is the vertical operator defined above.

#### 3.2 The Heat Kernel and the Propagator

With this generalized Laplacian, we can define  $K_t$ , the integral kernel for the operator  $e^{-tD}$ , living as a section of  $\text{Sym}^2(\mathcal{E})$ . Since the Laplacian splits as a sum, so does its kernel:  $K_t = K_t^\Delta + K_t^{\text{vert}} + K_t^{\text{matter}}$ .

In order to describe  $K_t$  it will be useful to decompose the classical BV complex  $\mathcal{E}$ . This complex is finite-dimensional as a dg-module over the ring  $C^\infty(\mathbb{R}^4)$ , so we can split it up, as a graded vector space, as

$$\mathcal{E} = C^\infty(\mathbb{R}^4) \otimes (\mathcal{Y} \otimes \mathfrak{g} \oplus \mathcal{S} \otimes V),$$

where  $\mathcal{Y}$  and  $\mathcal{S}$  are finite-dimensional graded vector spaces with a shifted symplectic pairing, and where the differential is independent of  $\mathfrak{g}$  and  $V$ . The heat kernel splits up according to the decomposition of  $\mathcal{Y} \oplus \mathcal{S}$  into BV particle species (i.e. into pairs  $(c, c^\vee), (A, A^\vee), (B, B^\vee), (\psi, \psi^\vee)$  of fields, which are paired with one another).

**Remark 3.1.** I'm going to write down the heat kernel and the propagator for abelian Yang-Mills theory – the algebra  $\mathfrak{g}$  and representation  $V$  go along for the ride. More precisely,  $K_t$  and  $P(\varepsilon, L)$  both can be written as sums of tensors that include factors of the form  $\mathfrak{g}^{\otimes 2}$  or  $V^{\otimes 2}$ . In the former case, we'll always just have the tensor  $\kappa$  representing the pairing on  $\mathfrak{g}$ , and in the latter the tensor  $\mu$  representing the pairing on  $V$  appearing in the action.

1. So let's consider the term  $K_t^\Delta$  associated to the ordinary Laplacian on the bosonic fields first. The heat kernel for the usual Laplacian acting on functions on  $\mathbb{R}^4$  has the form

$$k_t(x, y) = \frac{1}{(4\pi t)^2} e^{-|x-y|^2/4t}.$$

Using the pairing on the finite-dimensional graded vector space  $Y$ , the whole heat kernel has the form

$$K_t^\Delta(x, y) = k_t(x, y) \cdot (K_{AA^\vee} + K_{BB^\vee} + K_{cc^\vee})$$

where the components  $K_{AA^\vee}$ ,  $K_{BB^\vee}$ , and  $K_{cc^\vee}$  come from the different irreducible components of the symplectic pairing on  $\mathcal{Y}$ . Explicitly, we find

$$\begin{aligned} K_{AA^\vee} &= dx^j \otimes *dy^j + *dx^j \otimes dy^j, \\ K_{BB^\vee} &= -\frac{1}{2} (\sigma^I \otimes \sigma'^I + \sigma'^I \otimes \sigma^I), \\ K_{cc^\vee} &= -(\text{dvol}_x \otimes 1 + 1 \otimes \text{dvol}_y) \end{aligned}$$

where we sum over repeated indices as usual.

2. To deal with the term  $K_t^{\text{vert}}$  we can use a trick. Note that  $D_{\text{vert}}$  squares to zero and commutes with the usual Laplacian acting on forms. Thus, for a fixed field  $\varphi \in \mathcal{E}$  we have

$$e^{-t(\Delta_\Omega + D_{\text{vert}})} \varphi = e^{-t\Delta_\Omega} (1 - tD_{\text{vert}}) \varphi = e^{-t\Delta_\Omega} \varphi - te^{-t\Delta_\Omega} D_{\text{vert}} \varphi.$$

It follows that the second piece of the analytic heat kernel can be written as

$$K_t^{\text{vert}} = -t(D_{\text{vert}} \otimes 1) K_t^\Delta.$$

3. Finally, for the matter term, we use the same idea as in (1) above. That is, the heat kernel for the matter sector is

$$K_t^{\text{matter}} = k_t \cdot \frac{1}{2} (\psi^j \otimes \psi'^j + \psi'^j \otimes \psi^j).$$

Ok, now we can compute the propagator, which means hitting the heat kernel above with  $Q^{\text{GF}} \otimes 1$  and integrating. Applying the gauge-fixing operator changes which summands of  $\mathcal{E} \otimes \mathcal{E}$  will appear in the sum over pairs of particle species. We'll end up, again, with three summands in the pure gauge sector, and one summand in the matter sector.

**Lemma 3.2.** The propagator of Yang-Mills theory on  $\mathbb{R}^4$  takes the form

$$\begin{aligned} P(\varepsilon, L) &= \int_\varepsilon^L dt \left( \frac{\partial k_t}{\partial x^i}(x, y) (P_{AB}^i + P_{A^\vee c}^i) \right. \\ &\quad \left. + \frac{\partial^2 k_t}{\partial x^i \partial x^j} P_{AA}^{ij} \right. \\ &\quad \left. + \frac{\partial k_t}{\partial x^i} P_{\psi\psi}^i \right), \end{aligned}$$

with the three rows coming from the three summands of the heat kernel, where the  $P^i$  are some combinatorial tensors of the form

$$\begin{aligned} P_{AB}^i &= \sigma_x^{ij} \otimes dy^j + *(dx^i \sigma_x^{1j}) \otimes \sigma_y^{1j} \\ P_{A^\vee c}^i &= (1 \otimes *dy^i + *dx^i \otimes 1) \\ P_{AA}^{ij} &= 4(\delta^{ij} dx^\ell - \delta^{i\ell} dx^j) \otimes dy^\ell \\ \text{and } P_{\psi\psi}^i &= (\Gamma^i \psi^j) \otimes \psi'^j + \psi'^j \otimes (\Gamma^i \psi^j). \end{aligned}$$

The key thing to note from this is that in the Feynman rules, we'll be able to draw edges from  $A$  to  $B$ , from  $A^\vee$  to  $c$ , from  $A$  to  $A$ , and from  $\psi$  to  $\psi$ .

### 3.3 One-loop Feynman Diagrams

Ok, so let's conclude by finally talking about some specific Feynman diagrams. We now know everything that we need to know in order to calculate the weight of a Feynman graph (with the bare interaction): we need to know the propagator  $P(\varepsilon, L)$  as a section of  $\mathcal{E} \otimes \mathcal{E}$ , and we need to know the interaction, as an element of  $\mathcal{O}_{\text{loc}}(\mathcal{E})$  (in our case it's a purely cubic functional on  $\mathcal{E}$ ). The Feynman weight is calculated by contracting these expressions together according to the topology of the graph.

So what are the 1-loop Feynman diagrams that occur? All 1-loop Feynman diagrams in theories like this – with purely cubic interaction – are given by *wheels*:  $k$ -gons with external legs at each vertex. We can check the following in the case of Yang-Mills theory (it's a purely analytic calculation, proven using Wick's lemma type considerations).

**Lemma 3.3.** The weights associated to wheels of size  $k$  in Yang-Mills theory are convergent in the  $\varepsilon \rightarrow 0$  limit unless  $k = 2$ .

So only one shape of Feynman diagram contributes. There are two internal edges to which we assign the Yang-Mills propagator. Since the propagator splits up as a sum, we can write our weight as a sum over labellings of the diagram, where we attach a particular summand of the propagator to each edge, and a particular summand of the interaction to each vertex. The weights that occur take the following forms.

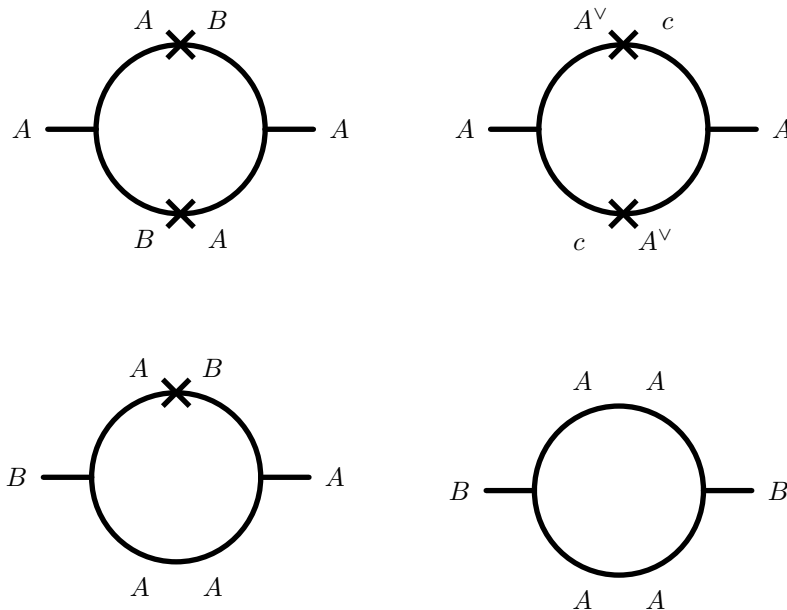


Figure 1: The four purely bosonic one-loop Feynman diagrams that contribute to the log divergence, and therefore to the one-loop  $\beta$ -function. The internal propagators are decorated when the species of a particle alters between its two end points.

We'll refer to the four diagrams in Figure 1 as diagram I to IV, or as  $\Gamma_I$  to  $\Gamma_{IV}$  (left to right, then top to bottom). We'll refer to the fermionic diagram in Figure 2 as diagram V, or as  $\Gamma_V$ .

**Remark 3.4.** There's actually one other possible diagram: a 1-loop correction to the fermion propagator. It turns

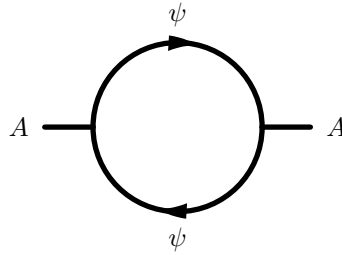


Figure 2: The remaining diagram that contributes to the one-loop  $\beta$ -function, depending on a choice of matter representation.

out that the weight of this diagram is exact for the classical BV differential, and therefore won't contribute to, for instance, the  $\beta$ -function.

Ok, so let's conclude by saying something about what it looks like to evaluate these diagrams. We have all the ingredients: for each diagram we need to take two summands of the propagator, and two summands of the interaction, then contract them together to obtain a local functional depending on  $\varepsilon$ . While we need the full expression to determine  $I[L]$  at one loop, to work out the one loop  $\beta$  function we only need the part asymptotic to  $\log \varepsilon$ . Finally, we must determine the cohomology class in  $\mathcal{O}_{\text{loc}}$  – in this case it turns out that the term corresponding to each diagram is individually closed (this might not always happen, it could be that only their sum is closed).

To actually work out the weight, we recall that  $\mathcal{E}$  can be factored as  $C^\infty(\mathbb{R}^4) \otimes (\mathcal{Y} \otimes \mathfrak{g} \oplus \mathcal{S} \otimes V)$ . When we're contracting together the various tensors we can write the result as a product of a term involving  $C^\infty(\mathbb{R}^4)$  (this is the term that involves doing an integral), a term involving  $\mathcal{Y}$  or  $\mathcal{S}$  (this term involves a little combinatorics), and a term involving  $\mathfrak{g}$  or  $V$  (this will just evaluate to the quadratic Casimir).

## References

- [Cos11] Kevin Costello. *Renormalization and effective field theory*. Number 170. American Mathematical Soc., 2011.
- [EWY18] Chris Elliott, Brian Williams, and Philsang Yoo. Asymptotic freedom in the BV formalism. *J. Geom. Phys.*, 123:246–283, 2018.

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