Abelian Duality for Generalised Maxwell Theories

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1 Duality of Quantum Field Theories

- What does the word "duality" mean in quantum field theory?
  - Roughly a duality is an *isomorphism* of quantum field theories.
  - There are several ways of talking about a quantum field theory. A perspective I like is a "Heisenberg"-type perspective where the fundamental objects are the *observables* in the theory, with additional structure describing *locality* of observables. This is nice because an observable is a natural object from a Lagrangian point of view: it's just a *functional* on the fields.
  - So from this point of view duality should "match up" the observables in two different quantum field theories, ideally in such a way that the locality structure was completely preserved.
  - Of course, we need more than this: so far I haven't invoked the "quantum" part of the quantum field theory at all. For each observable there should be a procedure for computing its quantum *expectation value*, for instance by evaluating an appropriate path integral. A duality between theories should be compatible with the expectation value map: an observable and its dual should have *equal* expectation values.
- Examples of dualities:
  - There are several conjectural dualities between theories we'd like to understand.
  - T-duality concerns 2d sigma models. It says there should be a duality between such a sigma model whose target has the structure of a torus fibration $E \to B$, and a sigma model whose target is the *dual* torus fibration. The T stands for "target".
  - S-duality concerns 4d gauge theories. It says there should be a duality between a gauge theory with gauge group G, and a gauge theory with the *Langlands dual* gauge group ${}^LG$. The theory should also invert the "gauge coupling constant". The S stands for "strong-weak".
  - In order to make these dualities work it may be necessary to introduce *supersymmetries*. For instance S-duality is only expected to exist for "maximally supersymmetric" ($N = 4$) gauge theories.
- The aim of this talk will be to make the idea of duality precise in a special case of the above examples where the theory becomes *free*. Free theories are mathematically much better understood than general interacting theories, so in these simple examples we can describe the duality explicitly and fully non-perturbatively.
- The free cases of the above examples are the sigma model with target just a torus (rather than a torus bundle), and gauge theories with gauge group a torus. These theories are actually closely related, as I'll explain.
- In these free theories we don't need to introduce any supersymmetry for duality to occur.
- Even in the free theories duality is still interesting. The dual of an observable can be quite different in nature to the original observable. Duality is closely related to a *Fourier transform*.

2 Examples of Theories: Generalised Maxwell Theories

- I'll describe our two fundamental examples in detail, and explain how they're related. In fact they fit into an infinite family of field theories in every dimension, which we expect to be related to one another in interesting ways under duality.
- From now on, all manifolds are compact, oriented and equipped with a Riemannian metric.
- The first example is the 2d sigma model with target a torus. For simplicity we'll take a rank one torus, i.e. a circle $\mathbb{R}/2\pi R\mathbb{Z}$ of radius R.
  - Let Σ be a 2-manifold. Fields are smooth maps $\phi\colon \Sigma \to \mathbb{R}/2\pi R\mathbb{Z}$.
  - Associated to ϕ we produce $d\phi$ by taking the derivative locally, and observing that we can glue. The kernel of this derivative map is the circle of constant functions, and the image consists of *closed integral 1-forms*, i.e. closed forms whose cohomology class lands in the lattice $H^1(\Sigma; 2\pi R\mathbb{Z}) \leq H^1_{dR}(\Sigma)$. This cohomology class represents the winding number type of ϕ.
  - The action is
  $$S_R(\phi) = \int_\Sigma d\phi \wedge *d\phi.$$
  Using the metric on Σ. We can rescale the circle to the circle of radius 1(and hence rescale the lattice to the standard one) at the expense of introducing a constant factor of R^2 in front of the action. This action defines a non-degenerate Gaussian on the space of 1-forms.
- The second example is the 4d gauge theory with gauge group a torus. Again, for simplicity take the torus to be rank one.
  - Let X be a 4-manifold. Field are connections A on principal circle bundles P.
  - Associated to A we produce F_A, its curvature. The kernel of the curvature map is the space of flat bundles, which is a finite rank torus (actually $H^1(X; S^1)$), and the image consists of *closed integral 2-forms*, i.e. closed forms whose cohomology class lands in the lattice $H^2(\Sigma; 2\pi R\mathbb{Z}) \leq H^2_{dR}(\Sigma)$. This cohomology class represents the first Chern class of P by Chern-Weil theory.
  - The action is
  $$S_R(A) = \int_X F_A \wedge *F_A.$$
  Again we can rescale the circle to a circle of radius 1 at the expense of introducing a constant factor of R^2 in front of the action.
- Without too much difficulty we can generalise this all to higher rank tori $\mathbb{R}^m/\mathcal{L}$. Now the curvatures will be closed 1- or 2-forms with coefficients in $\mathbb{R}^m$ with cohomology class landing in the $\mathcal{L}$-valued cohomology.
- These are on the surface very similar. To generalise we need to introduce some gadget with a "curvature" map to p-forms, where the kernel is a finite rank torus and the image consists of closed integral p-forms. This would allow us to define a completely analogous action functional.
- Such a gadget is provided by *differential cohomology*, using ideas developed by Dan Freed. We can define a classical field theory where the fields are given by degree p Deligne cocycles in much the same way as the above. To avoid getting bogged down we won't go into the details: it's still interesting just to think about the above two examples. But from now on I'll refer to the "p-form theory on an n-manifold" with the above examples corresponding to $p = 1, n = 2$ and $p = 2, n = 4$.

3 Defining and Computing Expectation Values

- That was purely classical field theory, so let's talk about some quantum data: expectation values of observables.
- This will be a machine that takes a functional on the global fields in a Lagrangian field theory (and maybe an extra piece of data too, that we'll discuss), and returns a *number*.

- We'll always talk about *polynomial* observables: that is polynomial functionals on the fields. If the fields are given by a vector space then this just means the completed symmetric algebra of the continuous dual space.
- This procedure uses a lot of ideas from the BV (Batalin-Vilkovisky) quantisation of free classical field theories. I learned this approach from Owen Gwilliam's thesis, and I think he discussed it at this seminar a few weeks ago. There are quite a few steps, so I'll just try to summarise the main ideas.
  - First we produce the *classical BV complex* of the classical field theory. This is an extension of the observables as polynomial functionals on fields to a cochain complex whose cohomology describes only the on-shell observables: functionals on the moduli space of classical solutions. This is done by taking the complex $PV(\Phi)$ of polyvector fields on the fields (in non-positive degrees), with differential given by contracting polyvector fields with the 1-form dS. So the degree zero part of the complex consists of "all" observables, and H^0 of the complex consists of on-shell observables (ignoring subtleties coming from gauge transformations).
  - Next we restrict to the "massive modes". Instead of building the classical BV complex of the whole theory, we work only with the subspace of the fields where the action is *non-degenerate*. In our examples this amounts to quotienting out by the finite rank torus which is the kernel of the curvature map. After we perform this restriction the classical BV complex becomes isomorphic to $\mathbb{R}$ since every critical point of a non-degenerate quadratic form is also a zero, so we're modelling functions on a contractible space.
  - Now, we'd like to *quantise* this classical BV complex. BV quantisation is a kind of deformation quantisation: we add onto the differential the *divergence* operator on polyvector fields (note that we're working non-perturbatively here, so we can use units where $\hbar = 1$). In the toy example where the space of fields is finite-dimensional, the resulting complex is isomorphic to a de Rham complex where the differential is twisted by the Gaussian e^{-S}, shifted down by the dimension of the space of fields.
  - We can compute the cohomology of this complex, and we find it's still isomorphic to $\mathbb{R}$ (trivialisable). We could discover this as an application of the homological perturbation lemma, but more concretely one uses the spectral sequence of a filtered complex, filtered by total degree (polyvector field degree plus polynomial degree for the coefficients), whose E^2-page is the classical BV complex.
  - So since the quantum BV complex is quasi-isomorphic to $\mathbb{R}$, there is an $\mathbb{R}$-torsor of linear maps from the global observables to $\mathbb{R}$. The *expectation value* is the unique such map sending 1 to 1.
  - Finally, we must recover the massless modes (the subspace of the fields which we quotiented out by to make the action nondegenerate). For each massless mode μ, we perform the above procedure for the observable $\mathcal{O}(\mu + -)$, thought of as a functional on the massive modes. The expectation value calculation described then gives us a section of a rank one local system over the moduli space of massless modes. In our examples, this moduli space will be compact (a finite rank torus), and the local system will be trivialisable, so we can integrate the resulting section to get a finite answer.
- I lied about one important point in the above construction: in the story as I described it one the cohomology of the classical BV complex is actually much too big. Roughly the reason is that there are many "distributional" observables which aren't hit by the classical BV differential. The fix is a procedure called *smearing*, which approximates these distributional observables by functional observables.
- For brevity I won't explain this procedure in detail, but I should point out that smearing involves a *choice*, roughly a choice of parametrix for a Laplace operator, and the expectation values depend on this choice. However, we can make this choice once-and-for-all at the start, and produce a coherent duality for all observables.

4 Constructing Dual Observables

- In order to explain how we construct dual observables, I have to say something about how we can actually *compute* expectation values of observables, and what they have to do with path integrals.
- Since (after choosing a smearing) the expectation value map is uniquely determined by being a linear map that vanishes on the image of the quantum BV differential, and that sends 1 to 1, any map we can find that satisfies these conditions will compute the correct number. We use a "regularisation" procedure to compute expectation values by genuine path integrals.

- Notice that the fields in our theories (at least the massive modes) are modelled by p-forms on a compact orientable manifold. This space admits a filtration by finite-dimensional spaces using *the spectrum of the Laplacian.* The idea is to do our calculations on each filtered piece (we say, at each regularised level) separately using an actual finite-dimensional path integral, then to take a limit.
- One checks that this procedure satisfies the required conditions, so compute the correct expectation value map. All our calculations of the "dual" of an observable work at each regularised level, and use simple algebraic and Fourier analytic manipulations of the finite-dimensional path integrals.
- The idea is to apply *Plancherel's theorem* to the path integral. Suppose we have an observable in the p-form theory that extends naturally to a functional on all p-forms (for instance, polynomials in 'wedge with something and integrate over a submanifold'). Then the path integral is an expression of the form
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D\alpha \, \mathcal{O}(\alpha) e^{-S(\alpha)} \delta^p_{cl,\mathbb{Z}}.$$
Let me explain what this expression means. The integral is over all p-forms α (up to some energy cutoff, which we suppress). We denote the partition function by Z: dividing through by it ensures the normalisation condition that 1 maps to 1 (and ensures that the expression converges as the cutoff goes to infinity). We abuse notation and write the action and the observable $\mathcal{O}$ as a function of the field strength α, rather than the field itself. Finally, $\delta^p_{cl,\mathbb{Z}}$ is the distribution that imposes the "closed and integral" condition to our p-forms.
- We apply Plancherel's theorem to this expression to find
$$\langle \mathcal{O} \rangle = \frac{1}{\widehat{Z}} \int D\hat{\alpha} \, \widehat{\mathcal{O}}(\hat{\alpha}) e^{-\hat{S}(\hat{\alpha})} \widehat{\delta^p_{cl,\mathbb{Z}}}.$$
The crucial manipulation here is to note that the Fourier transform of a polynomial times a Gaussian is a polynomial times the *dual* Gaussian. We call that dual polynomial $\widehat{\mathcal{O}}$.
- Using Hodge theory we can compute the Fourier transform of the distribution $\delta_{cl,\mathbb{Z}}$. After using the metric to identify $\Omega^p(X)$ with its dual, one computes $\widehat{\delta^p_{cl,\mathbb{Z}}} = *\delta^{n-p}_{cl,\mathbb{Z}}$, the pushforward under Hodge star.
- Thus, one finds
$$\langle \mathcal{O} \rangle = \frac{1}{\widehat{Z}} \int D\beta \, \widehat{\mathcal{O}}(*\beta) e^{-\hat{S}(*\beta)} \delta^{n-p}_{cl,\mathbb{Z}}$$
where the integral is an integral over $(n-p)$-forms. And we observe that *this* computes the expectation value of a *dual* observable $\widehat{\mathcal{O}}(*-)$ in a *dual* $(n-p)$-form theory, with a dual action $\hat{S}$ which is readily computed to be the action in the $(n-p)$-form theory with target the dual torus.

5 Local Structure of Duality

- To conclude, let me say a little bit about how I expect this to relate to *locality* of observables.
- Given an open set $U \subseteq X$, we can investigate the subalgebra of observables which depend only on the values of the fields on U. That is, assuming (as is the case in our examples), the global fields Φ are the global sections of a flasque sheaf of vector spaces on X, we have an inclusion $\mathrm{Sym}(\Phi(U)^\vee) \hookrightarrow \mathrm{Sym}(\Phi(X)^\vee)$. The natural question is: does duality preserve these subalgebras?
- In the calculations I can do, this is the case, and these calculations should be representative.
- An important example is the *Wilson and 't Hooft* operators in 4d abelian gauge theories. One computes that these are interchanged under duality, and both are local observables in a neighbourhood of a contour.
- I expect duality to provide an isomorphism of *factorisation algebras*, this language – as developed by Costello and Gwilliam – provides meaning to the idea of a local net of observables in a quantum field theory. The recipe I described gives a candidate observable dual to a local observable, but one must check that this observable is still local, and that not only are *global* divergences preserved by duality, but also *local* divergences on each open set. Understanding this behaviour is on-going work.