BV Quantisation of Higher Abelian Gauge Theories and Abelian Duality

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Goals

- There are many mathematical conjectures inspired by *S- and T-dualities* in physics. This work gives a complete mathematical axiomatisation of a simple family of examples.
- We describe *free* theories generalising abelian Yang-Mills theories, in all dimensions, and a duality relationship between *local observables*.
- This duality preserves the *expectation values* of local observables, i.e. the Feynman path integral.
- We can explicitly compute duals for many interesting examples of observables.
- The interesting observables are often *nonperturbative*, so our model of the quantum field theory must be able to deal with this. This is possible for free theories.
- The fields in dual theories are related by *Hodge star*, so a theory with p-form fields is dual to a theory with $(n-p)$-form fields. The abelian gauge group is also replaced by the dual torus.

Lagrangian Field Theories

- The data describing a field theory is a *sheaf of fields* and an *action functional*.
- Local *classical states* are given by the *derived critical locus* of the action, and *classical observables* by the algebra of functions on this space.
- There are often local symmetries (*gauge transformations*), and equivalent fields are not physically distinguishable. One way of encoding this is to describe fields by a *simplicial (abelian) group* where only π_0 is physically measurable.
- More practically, one often uses a *cochain complex* in degrees ≤ 0, which is equivalent by Dold-Kan.
- Main example: abelian Yang-Mills with gauge group V/Λ (*Maxwell's theory*). Fields U on are the complex
$$\Lambda[2] \hookrightarrow \Omega^0(U;V)[1] \to \Omega^1(U;V)$$
- More generally, for $0 < p < \dim U$ one generalises this to the complex
$$\Lambda[p] \hookrightarrow \Omega^0(U;V)[p-1] \to \cdots \to \Omega^{p-1}(U;V)$$
viewed as describing "higher" circle bundles with connection [Freed].
- In all these theories there's a *curvature map* to closed p-forms. The action is the L^2-norm squared of this form. We call these theories *generalised Maxwell theories*
- To encode the derived critical locus of an action functional, introduce the *classical Batalin-Vilkovisky formalism*:
  - Functions on the derived critical locus of S are described as *polyvector fields* on the space of fields, with a new differential ι_{dS}.
  - Note that even if S is not defined locally (as above, forms may not be L^2), its variation dS will be.
  - Alternatively, we can view this as deforming $\mathcal{O}(T^*[-1]\Phi(U))$, where $\Phi(U)$ are the local fields on an open set U.

Factorisation Algebras

- We want an axiomatic way of talking about the algebras of local observables in a field theory, along with the relationships between observables on different open sets. ([Gwilliam], [Costello-Gwilliam]).
- A *prefactorisation algebra* on X is a precosheaf of cochain complexes on X equipped with S_k-equivariant isomorphisms
$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_k) \to \mathcal{F}(U_1 \sqcup \cdots \sqcup U_k)$$
for every collection $U_1, \ldots, U_k \subseteq X$ of disjoint open sets.
- To be a *Factorisation algebra*, observables on an open set U need to all be "determined" by observables in small neighbourhoods of finitely many points.

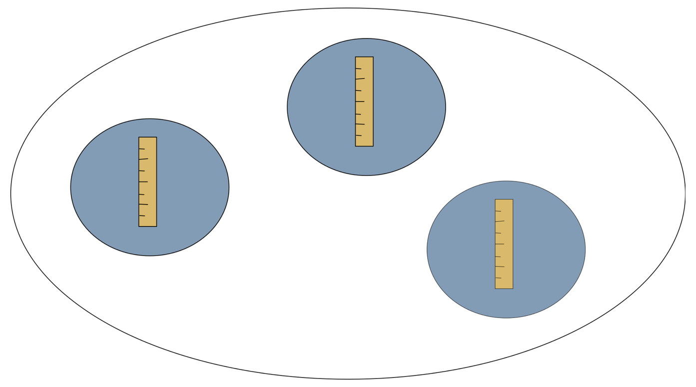

Figure : In a factorisation algebra, observables are determined by finitely many measuring implements of finitely small radius of sensitivity

- The classical BV formalism above produces examples of factorisation algebras.

BV Quantisation

- The theories that admits easy quantisations are the *free* theories. A theory is free if the classical differential ι_{dS} increases polynomial degrees by one. Generally this means the action functional is *quadratic*.
- This is natural if we're interested in *path integrals*. We might like to formally evaluate expressions like
$$\int_{H^0(\Phi)} \mathcal{O}(\phi)e^{-S(\phi)}D\phi$$
which is a *Gaussian integral* if the theory is free.
- **Example:** If V is a finite-dimensional vector space, S is a positive definite quadratic form on V and $f \in \mathcal{O}(V)$, the path integral $\int_V f(x)e^{-S(x)}dx$ can be computed as the cohomology class of the top form fdx in the twisted de Rham complex with differential $d-(\wedge dS)$. Contracting with the volume form dx turns this into the cohomology class of f in a complex of polyvector fields.
- In this picture, the quantum BV operator D is the image of d as a differential on polyvector fields. This still makes sense in infinite dimensions.
- There is a Poisson bracket on polyvector fields. The operator D is given by the Poisson bracket in degree 1: $D = \{,\}\colon T_0\Phi(U) \otimes \mathcal{O}(\Phi(U)) \to \mathcal{O}(\Phi(U))$, and extends to higher degrees by the formula
$$D(\phi \cdot \psi) = D(\phi)\cdot\psi + (-1)^{|\phi|}\phi \cdot D(\psi) + \{\phi, \psi\}.$$
- The *complex of quantum observables* is the result of adding D to the differential in the complex of classical observables. We can do this locally to get a factorisation algebra.

Expectation Values

- The above story suggests that the path integral for free theories should admits a nice homological description. It does in nice situations!
- If the complex of fields is a complex of *vector spaces*, and if e^{-S} is *nondegenerate*, then there is a canonical quasi-isomorphism from H^0 of the complex of quantum observables to $\mathbb{R}$.
- We can compute this using path integrals by filtering our fields by finite-dimensional vector space (*regularisation*). When the fields are given by p-forms on a compact manifold we can use the filtration by eigenvalues of the Laplacian.
- The fact that the map is canonical means any way of computing it gives the same answer. A nice method is using *Feynman diagrams*.
- Let $\mathcal{O}$ be a monomial observable. Write it as a product of linear observables
$$\mathcal{O} = \mathcal{O}_1^{n_1}\cdots\mathcal{O}_k^{n_k}.$$
- The expectation value is a sum over all graphs with k vertices of degrees n_i, with each graph weighted using the $\mathcal{O}_i$.

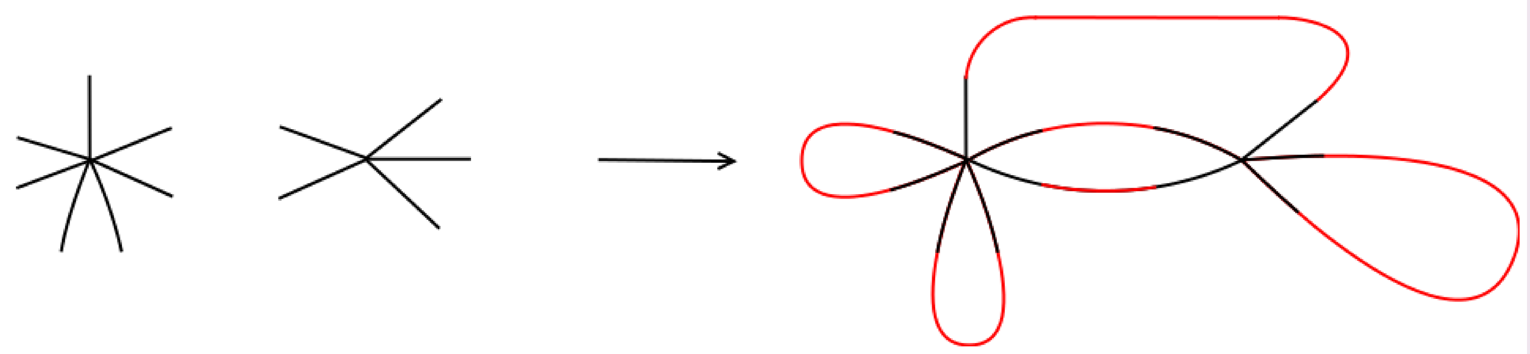

Figure : One of the diagrams in the computation of the expectation value of an observable $\mathcal{O}_1^7\mathcal{O}_2^5$.

- Edges between vertices $\mathcal{O}_i$ and $\mathcal{O}_j$ contribute weights of form $\int_X \mathcal{O}_i \wedge *\mathcal{O}_j$. The total weight of a diagram is the product of the weights of its edges.
- For this to give well-defined answers, we generally need to work with *smooth* (as opposed to distributional) observables; things like pairing a p-form with an $(n-p)$-form (or polynomials therein). General observables can be approximated by such smooth observables.
- We can compute expectation values in generalised Maxwell theories by working through the theory where fields are closed p-forms (like replacing connections by their curvatures). We need to modify the definition of the expectation value on spaces X with $H^p(X) \neq 0$, to deal with an *integral periods* condition.
- Specifically, we push a local observable forward to a global observable on X, then compute the path integral. We can do this by, instead of integrating over closed p-forms, we integrate over the lattice of closed p-forms with periods in the lattice Λ.

Fourier Duality

- $$(p - \text{form theories, gauge group } T) \overset{\text{Fourier duality}}{\longleftrightarrow} ((n-p) - \text{form theories, gauge group } T^\vee)$$
- First described in [Witten], [Verlinde].
- We'd like to define the Fourier dual of an observable using Feynman integrals, using the fact that the Fourier dual of a Gaussian polynomial is also a Gaussian polynomial. I.e.
$$\widetilde{\mathcal{O}}(\widetilde{\alpha})e^{-\widetilde{S}(\widetilde{\alpha})} = \int_{H^0(\Phi(U))} \mathcal{O}(\alpha)e^{-S(\alpha)+i\langle\alpha,\widetilde{\alpha}\rangle}D\alpha.$$
However, this doesn't work locally; the integrals don't converge unless U is compact.
- We fix this by defining the dual $\widetilde{\mathcal{O}}$ using *Feynman diagrams*, and checking that this agrees with the path integral when U is compact.
- The idea is a lot like Feynman diagrams for expectation values, but some edges can be left free (*source terms*).

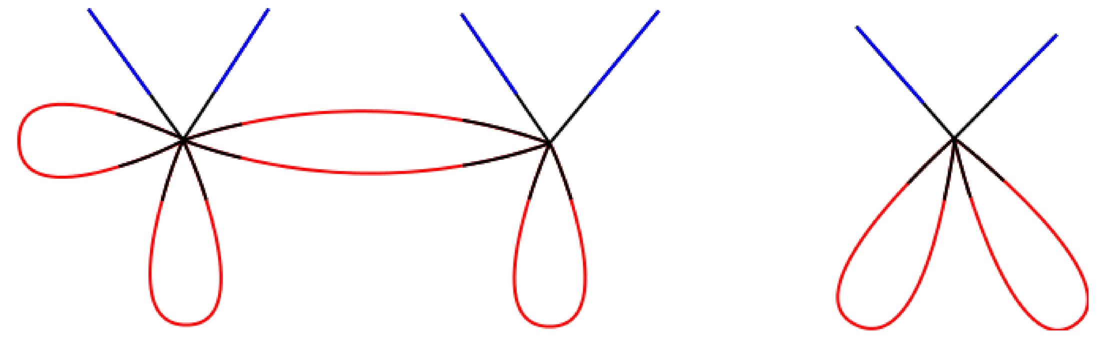

Figure : A term in the dual of $\mathcal{O}_1^8\mathcal{O}_2^6\mathcal{O}_3^6$ contributing a multiple of $(*\mathcal{O}_1^2)(*\mathcal{O}_2^2)(*\mathcal{O}_3^2)$

- Edges between vertices $\mathcal{O}_i$ and $\mathcal{O}_j$ contribute weights of form $\int_X \mathcal{O}_i \wedge *\mathcal{O}_j$. Source terms contribute weight i. The total weight of a diagram is the product of the weights of its edges.
- **Theorem:** An observable $\mathcal{O}$ and its Fourier dual $\widetilde{\mathcal{O}}$ in generalised Maxwell theories have the same expectation value.
- The idea of the proof is to use *Plancherel's theorem* applied to an observable $\mathcal{O}$ as a functional on all p-forms, and the delta distribution $\delta_{cl,\Lambda}$ on the *closed p-forms with periods in* Λ. This distribution is Fourier dual to its pushforward under Hodge star.
- The theorem can be restated as a *correspondence of factorisation algebras*. It's only a correspondence because we have to choose a way of extending $\mathcal{O}$ from closed p-forms to all p-forms.

Future Work

- This duality should extend naturally to supersymmetric abelian gauge theories, in particular the abelian $N=4$ theory.
- The $N=4$ theory admits a $\mathbb{CP}^1$ family of topological twists. Duality is supposed to exchange antipodal twists.
- Duality should give an equivalence on categories of D-branes, and the abelian equivalence should recover geometric class field theory à la Laumon-Rothstein, according to the work of Kapustin and Witten.

References

- More details are available in
  - Chris Elliott, Abelian Duality for Generalised Maxwell Theories, arxiv:1402.0890.
- Other references:
  - Kevin Costello and Owen Gwilliam, *Factorization Algebras in Quantum Field Theory*, 2013. book in progress.
  - Daniel Freed, Dirac Charge Quantization and Generalized Differential Cohomology, *Surveys in Differential Geometry*, VII:129194, 2000
  - Owen Gwilliam, *Factorization Algebras and Free Field Theory*, PhD thesis, Northwestern University, 2012
  - Erik Verlinde, Global aspects of Electric-Magnetic Duality. *Nuclear Physics B*, 455(1):211-225, 1995
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