Notes on "Localisation of $\mathfrak{g}$-modules"

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January 12, 2014

Let $\mathfrak{g}$ be a semisimple Lie algebra over a field $\Bbbk$. One is normally interested in representation of $\mathfrak{g}$, i.e. $\mathfrak{g}$-modules. If $\mathfrak{g}$ were commutative, one could use the tools of algebraic geometry. More precisely, for A a commutative ring, one has the equivalence of categories:

$$\{\text{A-modules}\} \leftrightarrow \{\text{quasi-coherent sheaves over } \mathrm{Spec}(A)\}\,.$$

However, the interesting cases of Lie algebras are noncommutative, so how is one going to build an algebro-geometric which captures the representation theory of $\mathfrak{g}$-modules? The answer is given in the work of Beilinson-Bernstein on localisation[1], and has essentially three parts.

(i) The first part is the definition of algebraic-geometric object associated to the Lie algebra of G called the *flag variety* X. This parametrises the Borel subalgebras, and to which weight of G we can construct G-equivariant line bundles over X.

(ii) The second is that one can consider differential operators on the above line bundle. These form a sheaf of so-called *twisted differential operators* over the flag variety. One can then using previous work due to Konstant describe the global section of these sheaves.

(iii) Finally, the flag variety is $\mathfrak{A}$-*affine*, which means that to understand the category of sheaves of $\mathfrak{A}$-modules is equivalent to understanding the category of modules for the ring $D_A \equiv \Gamma(X, \mathfrak{A})$, where $\mathfrak{A}$ will be the kind of t.d.o.s obtained in item (ii).

[1] One of the functors which realises the equivalence of categories above is called localisation. Namely, to an A-module M we associate the quasi-coherent sheaf $\widetilde{M}$, also denoted by $M \otimes_A \mathscr{O}_{\mathrm{Spec}(A)}$, over $\mathrm{Spec}(A)$ and conversely to any quasi-coherent sheaf $\mathscr{F}$ over $\mathrm{Spec}(A)$, the glocal sections $\Gamma(\mathrm{Spec}(A), \mathscr{F})$ form an A-module.

In these notes I will present the emphasised objects in the three items above. Then I'll explain Beilinson-Berstein result with some sketch proofs. To keep it relatively short I will not treat the applications which are very briefly sketched in their paper. I hope the material presented can give some familiarity of how to work with all the objects introduced. The reason for this choice is that probably the applications will appear very heavily in other talks in this seminar.

1 Flag variety

In [BB81], they consider G a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic 0. For simplicity we will consider G to be a connected semisimple algebraic group[2]. Recall that a Borel subgroup is defined as a maximally closed connected solvable subgroups of G, or equivalently, they are the minimal parabolic subgroups P of G[3]. We have the following results about Borel subgroups.

Proposition 1. *(1) Any two Borel subgroups B_1, B_2 are conjugate by an element $g \in G$.*

(2) The normalizer of B is B itself.

Let X denote the set of all Borel subgroups. From the above we can identify the set X with G/B, for B a fixed Borel subgroup of G. Since it is a general result that the quotient of an algebraic group G by a closed subgroup B is an algebraic variety, we obtain that X has the structure of an algebraic variety.

We list some of its properties without proof. For that we fix some notation. Let N (resp. N^-) denote the unipotent radical of B (resp. B^-, the conjugate Borel subgroup to B), $B/N = H$ is a maximal torus in G, and let $\mathfrak{h} = \mathrm{Lie}(H)$, $\mathfrak{b} = \mathrm{Lie}(B)$ and $\mathfrak{n} = \mathrm{Lie}(N)$ be the corresponding Lie algebras. Let Δ be the root system associated to B, and Δ^+ the set of positive roots[4]. Denote by W the Weyl group corresponding to this root data.

(i) The flag variety X has a cover by the affine open sets gN^-B/B, for $g \in G$. Each of these open subsets is isomorphic to $\Bbbk^{|\Delta^+|}$.

[2] Any reductive group can be realised as the product of a semisimple group and a torus.
[3] We say a subgroup P of G is *parabolic* if G/P is a complete variety.
[4] Recall $\Delta \equiv \{\lambda \in \mathfrak{h}^* | \mathfrak{g}_\lambda \neq 0\} \setminus 0$ and $\Delta^+ \subset \Delta$ is the subset of λ s.t. $\mathfrak{g}_\lambda \subset \mathfrak{b}$.

(ii) X is a projective variety

(iii) Let $X_w \equiv BwB$, for $w \in W$. We have a decomposition $X = \sqcup_w BwB$. And each X_w[5] is a closed submanifold of X, isomorphic to $\Bbbk^{\ell(w)}$, where $\ell(w)$ is the length of w. Moreover $\bar{X_w} = \cup_{y \leq w} X_y$, where we use the Bruhat ordering.

We will be interested in G-equivariant sheaves on X.

Definition 1. A sheaf $\mathscr{F}$ over X is G-equivariant if we are given a morphism of sheaves

$$\varphi : p^* \mathscr{F} \to \sigma^* \mathscr{F},$$

where $p : G \times X \to X$ is the natural projection and $\sigma : G \times X \to X$ is the action by conjugation. This is asked to satisfy some usual cocycle condition, i.e. that the two maps one can form, using p, σ and the multiplication of G, between sheaves over $G \times G \times X$ agree.

The sheaf morphism φ induces a map on sections from $p^*\mathscr{F}$ to $\sigma^*\mathscr{F}$, that is a map between $k[G] \otimes \Gamma(X, \mathscr{F}) \to k[G] \otimes \Gamma(X, \mathscr{F})$. If restricted to elements of the form $1 \otimes f$, for $f \in \Gamma(X, \mathscr{F})$ we obtain a map

$$\tilde{\varphi} : \Gamma(X, \mathscr{F}) \to k[G] \otimes \Gamma(X, \mathscr{F}).$$

This is equivalent to an action of G on $\Gamma(X, \mathscr{F})$. The same reasoning also gives an action of G on all cohomology $H^i(X, \mathscr{F})$. This construction of representation via sections of equivariant sheaves is very important in geometric representation theory.

Now let V be a G-equivariant vector bundle over X. The fiber V_B of V at $B \in X$ is a B-module[6]. Conversely, given a B-module U consider the action of B on $G \times U$ given as:

$$b \cdot (g, u) = (gb^{-1}, bu) \quad b \in B, g \in G, u \in U.$$

Then $V = B \backslash (G \times U)$ is a G-equivariant bundle with $V_B = U$.

[5] This is called a *Schubert cell* and will come up in other articles in this seminar. Its closure is known as *Schubert variety*.

[6] Indeed, one has a map for every $g \in G$ from V_B to $V_{g \cdot B}$, if $g \in B$, we know that $g \cdot B = B$ so it actually gives a map from B to $GL(V_B)$.

Remark. In particular, G-equivariant line bundles over X correspond to 1-dimensional representations of B. Moreover, N (the unipotent radical of B) acts trivially on 1-dimensional representations, i.e. a 1-dimensional representation of any unipontent element is trivial. So the representation corresponding to a G-equivariant line bundles is actually induced from a representation of B/N, since N acts trivially. Since $H = B/N$ is abelian, its representations are given by a character $\lambda \in \mathrm{Hom}(H, \mathbb{G}_m)$, so for any λ we get a G-equivariant line bundle on X which we denote $\mathscr{L}(\lambda)$.

The above should be enough to understand the construction used in [BB81]. We now will describe a little of how $\mathscr{D}$-modules allow to manage the noncommutativity of Lie algebras.

2 Twisted Differential Operators

We will define $\mathscr{D}$-modules in a particular way such that the case we are interested in sits naturally in this definition.

Definition 2. For an algebraic variety X, with structure sheaf $\mathscr{O}$ consider the sheaves $\mathscr{D}^n$ defined inductively on an open set U by

$$\mathscr{D}^n(U) \equiv \left\{\varphi \in \mathrm{End}\left(\mathscr{O}(U)\right) | [\varphi, f] \in \mathscr{D}^{n-1}(U),\ f \in \mathscr{O}(U)\right\},$$

where we pose $\mathscr{D}^0(U) = \mathscr{O}(U)$. The sheaf $\mathscr{D}_X$, which we will abreviate as $\mathscr{D}$, on an open set U is then just $\mathscr{D}(U) \equiv \cup_{n\geq 0}\mathscr{D}^n$. A *$\mathscr{D}$-module $\mathscr{M}$* is a sheaf of $\mathscr{O}$-modules with a map of sheaves $\mathscr{D} \to \mathrm{End}(\mathscr{M})$[7].

Remark. This definition gives a natural filtration to $\mathscr{D}$, which will be used later on. More importantly, to any locally free $\mathscr{O}$-module $\mathscr{F}$, replacing $\mathscr{O}$ by $\mathscr{F}$[8] in the above construction we obtain a sheaf $\mathscr{D}^{\mathscr{F}}$ which is evidently a $\mathscr{D}$-module.

Remark. In particular, for line bundles $\mathscr{L}$ the sheaf $\mathscr{D}^{\mathscr{L}}$ obtained is an example of *twisted differential operators*. One can check this agrees with [BB81], as $\mathscr{O}$ sits naturally inside it, i.e. they are the elements of degree 0 with respect to the natural filtration.

[7] Here $\mathrm{End}(\mathscr{M})$ is the sheaf of internal hom from $\mathscr{M}$ to itself.
[8] Still taking $\mathscr{D}^0 = \mathscr{O}$, just change where the endomorphisms take place.

Example. For any $\lambda \in \mathrm{Hom}(H, \mathbb{G}_m)$ let $\mathscr{L}(\lambda)$ be the associated line bundle, we call $\mathscr{D}_\lambda = \mathscr{D}^{\mathscr{L}(\lambda)}$. More generally [BB81] define $\mathscr{D}_\lambda$ for any weight $\lambda \in P$[9] the $\mathscr{D}_\lambda$ module as follows. Let $\mathscr{U} \equiv U \otimes_{\Bbbk} \mathscr{O}$ and $\mathscr{J}_\lambda$ be the ideal generated by $\xi - \lambda(\xi)$ where we still denote by $\lambda : \mathfrak{b} \otimes_{\Bbbk} \mathscr{O} \to \Bbbk$ the map induced by λ on the subsheaf $\mathfrak{b} \otimes_{\Bbbk} \mathscr{O}$, and ξ is a local section of $\mathscr{U}$. Hence $\mathscr{D}_\lambda \equiv \mathscr{U} / \mathscr{J}_\lambda$.

3 Global sections of $\mathscr{D}_\lambda$

Since we want to study the representation theory of G, a good first step is to study the representation theory of $\mathfrak{g}$. The latter is actually equivalent to the representation theory of $U(\mathfrak{g})$, which henceforth we denote only by U.

The important result which makes the generalisation of the geometric idea of localisation possible is a relation between certain U-modules and $\Gamma(X, \mathscr{D}_\lambda)$-modules. To describe it we first need a map from U to $\Gamma(X, \mathscr{D}_\lambda)$, this is obtained as follows.

From the action of G on X, we have an induced map $\mathfrak{g} \to T_X$, from the Lie algebra to the tangent bundle of X. One also has a map $\mathfrak{g} \to \Gamma(X, \mathscr{D}_\lambda)$ constructed in the same way one does the induced map. Namely, given $a \in \mathfrak{g}$ we consider $\partial_a \in \mathscr{D}_\lambda$ defined on a section $s \in \Gamma(X, \mathscr{L}(\lambda))$ by

$$\varphi\left((a \otimes 1) \cdot \varphi^{-1}(\sigma^* s)\right) = \sigma^*(\partial_a s).$$

Actually this formula makes sense for a any differential operator on G, not only right-invariant vector fields, hence the map extends to U, remember that U can be seen as the right-invariant differential operators on G.

We call this map $\Phi_\lambda : U \to \Gamma(X, \mathscr{D}_\lambda)$. One of the main results which allows Beilinson-Berstein ideas to work is the following, according to Dixmier due to Konstant.

Before enoucing the theorem we need some notation. Let $Z \subset U$ be the center of the universal enveloping algebra. We call χ_λ the central character of Z associated to λ. We recall this construction. Any $\lambda \in P$ induces a map $\lambda : \mathfrak{h} \to \Bbbk$. Then by the Poincaré-Birkhoff-Witt theorem, the universal enveloping algebra of $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ splits as $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus W$, for some subalgebra W. Hence the map λ gives a map $\chi_\lambda : U(\mathfrak{h}) \to \Bbbk$, because $\mathfrak{h}$ is commutative, so $U(\mathfrak{h}) \simeq \mathrm{Sym}(\mathfrak{h})$. The central character comes from noting

[9] We denote by P the weight lattice, i.e. $P = \{\lambda \in \mathfrak{h}^* | \lambda(\alpha^\vee) \ \ \alpha \in \Delta \in \mathbb{Z}\}$, and we denote by $\alpha^\vee$ the coroot associated to α.

that $Z = U^G$ (?), the invariant element of the universal algebra under the adjoint action of G, and composing χ_λ with the map $U^G \to U(\mathfrak{h})$ (?). We still use the same notation for the map $\chi_\lambda : Z \to \Bbbk$.

Theorem 1. *The map Φ_λ is surjective. Moreover, for any $z \in Z \subset U$ $\Phi(z) = \chi_\lambda(z) id$ and $ker \Phi_\lambda = U \cdot (ker \chi_\lambda)$.*

We will not fully prove this result as it is a little bit technical we will try however to give an insight of why this is true by looking at the commutative part of the statement.

Sketch. The associated graded to U is just the symmetric algebra over $\mathfrak{g}$. It is not hard using the canonical filtration of $\mathscr{D}_\lambda$ to see that the map Φ_λ actually descends to a map between the associated graded parts

$$\mathrm{gr}\Phi_\lambda : \mathrm{Sym}(\mathfrak{g}) \to \Gamma(X, \mathrm{gr}(\mathscr{D}_\lambda)).$$

Remark that $\Gamma(X, \mathrm{gr}(\mathscr{D}_\lambda)) \simeq \Gamma(X, \mathrm{gr}(\mathscr{D})) \simeq \Gamma(T^*X, \mathscr{O}_{T^*X})$. So composing with these isomorphisms we get a map $\widetilde{\Phi_\lambda} : \mathrm{Sym}(\mathfrak{g}) \to \Gamma(T^*X, \mathscr{O}_{T^*X})$.

Now recall, by a theorem of Chevalley $\Bbbk[\mathfrak{g}]^G \simeq \Bbbk[\mathfrak{h}]^W$ which gives a map $\sigma : \mathfrak{g}/G \to \mathfrak{h}/W$, where G acts on $\mathfrak{g}$ by the adjoint action and W acts on $\mathfrak{h}$ as usual. Hence one can define the nilpotent cone inside $\mathscr{N} \subset \mathfrak{g}$ by precomposing σ with $\mathfrak{g} \to \mathfrak{g}/G$ and taking the inverse image of $\bar{0}$, the orbit through 0. Then the celebrated theorem by Konstant says that $\mathscr{N}$ is a normal reduced subvariety of $\mathfrak{g}$. Nevertheless, $\mathscr{N}$ is singular. However, one has a nice resolution of singularities by the algebraic variety T^*X[10]. One then has a map

$$\gamma : T^*X \xrightarrow{\gamma'} \mathscr{N} \xrightarrow{\gamma''} \mathfrak{g} \simeq \mathfrak{g}^*,$$

where the last identification is using the Killing form. It is not hard to see as well that this corresponds to the moment map, induced from the action of G on X. [11]

[10] This is the Springer resolution, cf. Chris talk.

[11] Indeed, we can view a point of the cotangent bundle of the flag variety T^*X to be (b, λ), where b is a Borel subalgebra and $\lambda \in \mathfrak{b}^*$. There is an action of G on T^*X, by the adjoint action on the first element and the coadjoint action on the second. This is an action compatible with the canonical symplectic structure of T^*X and the associated moment map $\mu : T^*X \to \mathfrak{g}^*$ is $\mu(b, \lambda) = \lambda$. The nilpotent cone $\mathscr{N}$ is just the image of this morphism. Post composing with the Killing map we get the map γ we had in the text.

The crucial observation is that the map on the algebra of functions induced by γ coincides with the map Φ_λ, for $\chi = \chi_\lambda$. Now the fact that γ is surjective follows from: (i) γ'^* is an isomorphism on the algebra of functions, since γ' is a resolution of singularities; and (ii) γ''^* is a surjection, since $\mathscr{N}$ is a closed subvariety of $\mathfrak{g}$.

To indentify the kernel of this map one has to look more closely to how Konstant's result is proved. In this commutative part it should be given by $\mathrm{Sym}(\mathfrak{g})^G_+$, i.e. the non-constant elements of symmetric algebra of $\mathfrak{g}$ invariant under G.

The extension of the result for the whole associative algebras is an argument inducing on the filtrations.

We will not prove why the elements $z \in Z$ are mapped to $\chi_\lambda \mathrm{id}$, we refer the reader to [THT07] for a clear and detailed account of this calculation. $\square$

The important corollary we get from the above theorem is that the category of U-modules which act by the central character χ_λ is equivalent to the category of $\Gamma(X, D_\lambda)$-modules.

4 X is $\mathscr{D}_\lambda$-affine

The map we obtained in Konstant's theorem relates to $\Gamma(X, \mathscr{D}_\lambda)$ not of $\mathscr{D}_\lambda$ itself. So to U-modules one associates $\Gamma(X, \mathscr{D}_\lambda)$-modules and not sheaves over X. However for the flag variety these turn out to determine each other. To make this precise we define $\mathscr{D}$-affine varieties.

Definition 3. Let $\mathfrak{A}$ be a twisted differential operator on a variety X. We say X is $\mathfrak{A}$-affine if for every $\mathscr{O}$-quasi-coherent $\mathfrak{A}$-module $\mathscr{F}$

1. $\mathscr{F}$ is generated by the global sections $\Gamma(X, \mathscr{F})$;

2. the cohomology sheaves $H^i(X, \mathscr{F})$ vanish, for all $i > 0$.

The above gives an analogous result to the localisation we have with quasi-coherent $\mathscr{O}$-modules. The following theorem is sometimes called Morita theorem.

Theorem 2. *Denote by Mod($\mathfrak{A}$) the category of $\mathscr{O}$-quasi-coherent $\mathfrak{A}$-modules. If X is $\mathfrak{A}$-affine, then Mod($\mathfrak{A}$) is equivalent to Mod(A), where $A \equiv \Gamma(X, \mathfrak{A})$.*

Proof. The functors that realise the equivalence above are: $\Gamma : \text{Mod}(A) \to \text{Mod}(\mathfrak{A})$, $\Gamma(X, \mathscr{N}) = N$ taking global sections and $\Delta(N) = \mathfrak{A} \otimes_A N$ called the localisation functor.

We have to check that $\alpha = \Gamma \circ \Delta$ is an isomorphism. Let $M \in \text{Mod}(A)$, the functor Δ is right exact, and since X is $\mathfrak{A}$-affine, Γ is exact by condition 2. So the functor α is right exact. So let

$$A^J \to A^I \to M \to 0,$$

be an exact sequence. Applying α we obtain

$$A^J \to A^I \to \alpha(M) \to 0,$$

since $\alpha(A) \simeq A$. This implies $\alpha(M) \simeq M$.

We need to check the other direction, i.e. that $\beta = \Delta \circ \Gamma$ is an isomorphism. Let $\mathscr{M}_0$ be the image of β in $\mathscr{M}$, $\mathscr{M}_0$ is an $\mathscr{O}$-submodule of $\mathscr{M}$, hence we have the exact sequence

$$0 \to \mathscr{M}_0 \to \mathscr{M} \to \mathscr{M}/\mathscr{M}_0 \to 0,$$

by taking global sections we get that $\Gamma(\mathscr{M}/\mathscr{M}_0) = 0$, since all its higher cohomology vanishes this gives $\mathscr{M}_0 = \mathscr{M}$. Now let $\mathscr{K}$ be the kernel of the map $\beta(\mathscr{M}) \to \mathscr{M}$, i.e.

$$0 \to \mathscr{K} \to \beta(\mathscr{M}) \to \mathscr{M} \to 0,$$

is an exact sequence, condition 1. says that β is surjective. We apply global sections to obtain

$$0 \to \Gamma(\mathscr{K}) \to \Gamma(\mathscr{M}) \to \Gamma(\mathscr{M}) \to 0,$$

where in the middle we used that α is an isomorphism, we get $\Gamma(\mathscr{K}) = 0$, thus β is an isomorphism and we are done. $\square$

Remark. Sometimes one define $\mathfrak{A}$-affine as the two categories being equivalent and then the above definition becomes a criterion.

Hence if one can proof that X is $\mathscr{D}_\lambda$-affine for all $\lambda \in P$ then we got a complete geometric characterisation of U-modules with a given central character χ_λ, that is, they are equivalent to the category of $\mathscr{D}_\lambda$-modules over X.

Indeed that is the case and this is the third part of [BB81] work.

Theorem 3. *For $\lambda \in P$, such that*

$$\lambda(\alpha^\vee) \notin \mathbb{N}, \quad \textit{for all}\, \alpha \in \Delta^+ \tag{1}$$

the flag variety X is $\mathscr{D}_\lambda$-affine.

Sketch. There are two things to check: (1) that the cohomology of any $\mathscr{F} \in \mathrm{Mod}(\mathscr{D}_\lambda)$ vanishes and (2) that $\mathscr{F}$ is generated by global sections.

We will first consider (1). The idea is as in any cohomological calculation one needs to transform the arbitrary sheaf $\mathscr{F}$ in a more tractable one, i.e. one for which the cohomology is known, and keep track of its cohomology through this procedure.

Consider V any finite dimensional G-module, as we explained there is an associated G-equivariant vector bundle $\mathscr{V}$ over X. We can filter it by G-equivariant vector bundles $\{\mathscr{V}_i\}_I$ the following way:

$$0 \subset \mathscr{V}_1 \subset \cdots \subset \mathscr{V}_k = \mathscr{V},$$

such that $\mathscr{V}_i/\mathscr{V}_{i-1} \simeq \mathscr{L}(\nu_i)$ for some $\nu_i \in P$, and moreover $\nu_1 \leq \cdots \leq \nu_k$. Now let $\mathscr{F}$ be any $\mathscr{D}_\lambda$-module one get a map

$$i_{\mathscr{F}} : \mathscr{F} \to \mathscr{V} \otimes \mathscr{F}(-\nu_1),$$

induced from $\mathscr{O} = \mathscr{V}_1(-\nu_1) \to \mathscr{V}(-\nu_1)$. Here we denote $\mathscr{F}(\mu) \equiv \mathscr{F} \otimes \mathscr{L}(\mu)$. The technical part is the following:

Claim. *When λ is dominant, i.e. satisfy (1). Then $i_{\mathscr{F}}$ has a right inverse $j_{\mathscr{F}}$.*

To finish the proof we consider $\mathscr{G}$ a quasi-coherent $\mathscr{O}$-module, and let $\varphi : \mathscr{G} \to \mathscr{F}$ be any $\mathscr{O}$-module morphism. We can factor it through $\mathscr{V} \otimes \mathscr{G}(-\nu_1) \to \mathscr{V} \otimes \mathscr{F}(-\nu_1)$. This gives induced maps on cohomology, since $H^i(X, \mathscr{V} \otimes \mathscr{G}(-\nu_1)) \simeq V \otimes_{\Bbbk} H^i(X, \mathscr{G}(-\nu_1))$. Now since V was arbitrary, we can choose it such that $-\nu_1$ is sufficiently large, then by Borel-Weil-Bott theorem, i.e. for $\mu \in P$ ($\mu(\alpha^\vee) < 0$ for $\alpha \in \Delta^+$) the associated line bundle $\mathscr{L}(\mu)$ is ample[12], we have that $H^i(X, \mathscr{G} \otimes \mathscr{L}(-\nu_1)) = 0$. As $\mathscr{G}$ was arbitrary we obtain that $H^i(X, \mathscr{F})$ should vanish for all $i > 0$. □

[12]Recall that this means that its positive cohomology when tensored with any $\mathscr{O}$-quasi-coherent sheaf vanishes, if twisted enough times. And that $\mathscr{L}(\mu)^m \simeq \mathscr{L}(m\mu)$.

Remark. We will just say a few words about how to prove the claim. Note that

$$\mathscr{V}_1 \otimes \mathscr{F}(\mu) \simeq \mathscr{F}(\nu_1 + \mu),$$

is a $\mathscr{D}_{\lambda+\nu_1+\mu}$-module. One just need to check that there is a projection from $\mathscr{V} \otimes \mathscr{F}(\mu)$ to $\mathscr{V}_1 \otimes \mathscr{F}(\mu)$. This follows from the action of Z on $\mathscr{V} \otimes \mathscr{F}(\mu)$, i.e., it acts by characters and $\mathscr{V}_1 \otimes \mathscr{F}(\mu)$ is an eigensheaf by the Harish-Chandra isomorphism. Indeed if it were not, since $Z \simeq U(\mathfrak{h})^W$, ν_1 could not be fixed by W, but in our decomposition ν_1 is the lowest weight, and since λ is dominant $w(\lambda) \geq \lambda$ for any $w \in W$, which implies that $\lambda + \nu_1 + \mu$ can not be conjugate (related by an element of the Weyl group) to any $\lambda + \nu_i + \mu$ for $i \neq 1$, when $\mu = -\lambda_1$ which was the case for $i_{\mathscr{F}}$.

One could write down what all these objects are for the case of $SL_2(\mathbb{C})$, we refer to [THT07] where this is done throughout Chapters 9, 10 and 11.

References

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