Framing Anomalies for Topological AKSZ Theories

Chris Elliott

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Plan for Today

- Start by explaining the Batalin–Vilkovisky (BV) formalism for perturbative QFT.
- Using this language, we can explicitly quantize Kapustin and Witten's family of twisted supersymmetric 4d gauge theories.
- Conclude by discussing framing anomalies in this language.

I'm going to discuss joint work with Owen Gwilliam and Brian Williams (including some work in progress).

The Classical BV Formalism

I'd like to start by discussing a formalism for thinking about classical field theory that is particularly appealing to mathematicians. We can package the data of a field theory, including (higher) gauge transformations purely algebraically.

Definition

A classical BV theory on a manifold M is a graded vector bundle $E \to M$ whose sheaf of sections we'll denote $\mathcal{E}$, together with

- A dg Lie structure on $\mathcal{E}$.
- A symplectic pairing $\mathcal{E} \otimes \mathcal{E} \to \mathrm{Dens}_M[-3]$ of degree -3.

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- A symplectic pairing $\omega \colon \mathcal{E} \otimes \mathcal{E} \to \mathrm{Dens}_M[-3]$ of degree -3.

To translate this into more familiar physical language, the usual fields are sections of E of degree 1, and the action functional is

$$S(\alpha) = \int_M \omega(\alpha, \frac{1}{2}\alpha + \frac{1}{6}[\alpha, \alpha]).$$

If $\alpha = \alpha_0 + \alpha_1$ is non-homogeneous, this expression includes the infinitesimal gauge symmetry action of α_0 on α_1.

Remark: This definition only includes cubic interactions, but it can be generalized to include higher order terms.

Example 1: Chern–Simons Theory

Let M be a compact oriented 3-manifold, and let $\mathfrak{g}$ be a semisimple Lie algebra. Suppose $\mathcal{E} = \Omega^\bullet(M) \otimes \mathfrak{g}$. This has a dg Lie structure, and a symplectic pairing via

$$\omega(\alpha, \beta) = \langle \alpha \wedge \beta \rangle.$$

If α is a degree 1 element, then the recipe above gives the Chern–Simons action functional

$$S(\alpha) = \int_M \langle \alpha \wedge \left(\frac{1}{2}\alpha + \frac{1}{6}[\alpha \wedge \alpha] \right) \rangle.$$

Topological AKSZ Theories

Let me generalize this example a bit. Now we will allow M to be an oriented n-manifold, for any n. We will replace $\mathfrak{g}$ by any dg Lie algebra L, with a non-degenerate invariant pairing of degree $3-n$.

Example: If $\mathfrak{g}$ is any Lie algebra, let $L = \mathfrak{g} \ltimes \mathfrak{g}^*[n-3]$.

Definition
The topological AKSZ theory on M with target BL is the classical BV theory associated to $\Omega^\bullet(M) \otimes L$, with pairing induced from the pairing on L.

The example of $L = \mathfrak{g} \ltimes \mathfrak{g}^*[n-3]$ is usually called BF theory. If we denote a generic field as (A, B), the action functional looks like

$$\int_M \langle B \wedge F_A \rangle.$$

Example 2: Kapustin–Witten Theory

The following example arises from $\mathcal{N} = 4$ super Yang–Mills theory on $\mathbb{R}^4$.

Theorem (E–Yoo, E–Safronov–Williams)

All twists of $\mathcal{N} = 4$ super Yang–Mills theory on $\mathbb{R}^4$ occur in families of the following type. We define a family of classical BV theories on $\mathbb{R}^4$ parameterized by the space $\mathbb{C}^3_{t_1,t_2,u}$ by

$$\mathcal{E}_{t_1,t_2,u} = \Omega^{\bullet,\bullet}(\mathbb{C}^2) \otimes \mathfrak{g}[\varepsilon],$$

where ε is a formal parameter of degree -1, with differential

$$\mathrm{d}_{t_1,t_2,u} = \overline{\partial} + t_1\partial_{z_1} + t_2\partial_{z_2} + u\frac{\mathrm{d}}{\mathrm{d}\varepsilon}.$$

This is a topological AKSZ theory if $t_1 = t_2 = 1$.

Comments on Quantization

Let us make a few nice observations about quantization.

- We say a classical BV theory $\mathcal{E}$ is of cotangent type if $\mathcal{E} = T^*[-3]\mathcal{B} = \mathcal{B} \ltimes \mathcal{B}^*[-3]$. In cotangent type theories, the only non-trivial Feynman weights have at most one loop! This applies to out Kapustin–Witten twists if $u = 0$.
- We say a classical BV theory on $\mathbb{C}^d$ is holomorphic if $\mathcal{E}$ is equivalent the sheaf of sections of a holomorphic vector bundle, and the dg Lie structure is described by holomorphic differential operators. In holomorphic theories, by a theorem of Williams one can construct a family of effective action functionals with no counterterms. All the theories we've been discussing today are holomorphic.

Theorem (E–Gwilliam–Williams)

The family $\mathcal{E}_{t_1,t_2,u}$ of Kapustin–Witten twisted theories admits a one-loop exact quantization to a family of quantum field theories over $\mathbb{C}^3$.

Given what we've discussed, the content of this theorem involves checking that there is no one-loop anomaly if $u = 0$, then checking that this quantization extends across to the full 3d family.

Oriented TQFT

Let me conclude by talking about framing anomalies, meaning – for me – obstructions to extending a quantization over $\mathbb{R}^n$ to a theory on a general oriented n-manifold.

In mathematics, we like to think of this in terms of an idea called factorization homology. There's something fairly precise that we can say, but I'll state it more informally.

Theorem (E–Safronov)

Given a topological quantum field theory on $\mathbb{R}^n$ with an action of $\mathrm{SO}(n)$, we can compute the algebra of observables on any smooth oriented n-manifold if the infinitesimal $\mathfrak{so}(n)$-action can be homotopically trivialized.

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What does this mean? Well, suppose our $\mathfrak{so}(n)$ action is Hamiltonian, so that it is generated by a current $J\colon \mathfrak{so}(n) \to \mathrm{Obs}(\mathbb{R}^n)$. A homotopy trivialization is just a potential for this current, i.e. a functional $\Theta\colon \mathfrak{so}(n) \to \mathrm{Obs}(\mathbb{R}^n)[1]$ of one degree lower so that

$$Q\Theta = J,$$

where Q is the differential on the complex $\mathrm{Obs}(\mathbb{R}^n)$.

Orienting Topological AKSZ Theories

So, to conclude our story, let's come back to topological AKSZ theories, say on $\mathbb{R}^n$. Classically, there is always a homotopy trivialization for the $\mathfrak{so}(n)$-action: this action is Hamiltonian with potential

$$J(X) = \int \langle \alpha \wedge \mathcal{L}_X \beta \rangle,$$

and because the differential on classical observables is generated by the de Rham differential, there is a potential via Cartan's formula:

$$\Theta(X) = \int \langle \alpha \wedge \iota_X \beta \rangle.$$

In general, however, there is an anomaly preventing us from lifting this potential to the quantum level.

Theorem (E–Gwilliam, in progress)

Let $\mathcal{E}_L$ be a topological AKSZ theory on $\mathbb{R}^n$ with target dg Lie algebra L. The framing anomaly is a class in

$$\bigoplus_{i+j=n,\, i>0} \mathrm{H}^i(\mathfrak{so}(n)) \otimes \mathrm{H}^j_{\mathrm{red}}(L).$$

If we wanted to ask for something stronger: for the homotopy trivialization at the quantum level to itself be Hamiltonian, then the corresponding obstruction would live in the non-reduced cohomology of L.

Thanks for listening!