Chris Elliott – Research Statement

My research takes place on the boundary between pure mathematics and theoretical physics. Some of the key ideas in modern high-energy physics – coming from the study of supersymmetric quantum field theories – have been strikingly influential in representation theory, algebraic geometry, and differential geometry. A famous example is how Seiberg and Witten's work on supersymmetric gauge theory led to the theory of Seiberg–Witten invariants, which revolutionized 4-manifold theory. However, many of these influential physical constructions do not stand on firm mathematical ground, meaning that they can only be applied on a heuristic or ad-hoc basis. The long term goal of my research program is to develop supersymmetric quantum field theory (QFT) into its own discipline of study within pure mathematics. Even glimpses of this mathematical theory have already had significant repercussions in algebraic and differential geometry, and in representation theory.

While ambitious, the goal of bringing supersymmetric QFT into mathematics is genuinely achievable. I have been including the word "supersymmetric" very deliberately. Although the development of a complete formalism for quantum field theory still seems out of reach for mathematicians, more progress can be made when we restrict attention to the class of supersymmetric theories, which carry a great deal of extra structure. For example, we can work with those aspects of a supersymmetric theory that are fixed by a choice of symmetry (the output of this procedure is usually called a "twist"). Studying these twisted theories is much more approachable using modern mathematical machinery, but the output still retains all the information useful in pure mathematics. Indeed, in joint work with Philsang Yoo, I have shown how the new language of derived algebraic geometry allows one to cast into a sharp, mathematical form a proposal of Witten and Kapustin, which relates certain supersymmetric 4-dimensional gauge theories to the geometric Langlands program, an endeavor at the heart of contemporary representation theory.

There is a lot of potential for my research program to involve undergraduate researchers in ways that are accessible to students, but still meaningful and important. I am excited to involve undergraduates in my research program in a significant way going forward. When reduced down to their basic ingredients, many of the calculations that I do can be distilled down to computations in linear algebra and multivariate calculus, and so it is possible to involve students in these calculations while introducing them to the more conceptual ingredients gradually, and in a way appropriate for their background. For example, some of the projects I will discuss below would allow me to teach students ideas from homological algebra and from Lie theory as we progress. It would also be very valuable to work with student researchers with expertise in programming, or with the use of computer algebra systems, in order to extend some calculations that I've performed by hand to a larger family of examples. In the more detailed discussion below I will highlight some previous work that I have done with REU students, and describe some plans for undergraduate research that tie in to my ongoing research.

My own work is inherently very interdisciplinary, including ideas coming from representation theory, homological algebra, homotopy theory, algebraic and differential geometry and higher category theory. A lot of these areas are unified by a cluster of ideas broadly known as "higher algebra", which is an important part of my approach to quantum field theory. I am able to engage with mathematics coming from a wide range of different fields, and as a result engage students whose interests lie in a wide range of directions. Even going beyond pure mathematics, collaboration and communication with theoretical physicists is also an essential part of this work, and has significantly furthered my research program so far. When I was a post-doc at the IHÉS I worked with the physicist Vasily Pestun, studying integrable systems related to 3d and 4d supersymmetric gauge theory [EP19]. I find it very rewarding to stay active as an organizer in the broader research community, and to this end my research goals related to QFT have allowed me to serve as a kind of ambassador between mathematicians and physicists, organizing many conferences, as well as smaller scale seminars, with the explicit goal of creating a dialogue between mathematicians and physicists whose interests overlap.

1 Algebra – Supersymmetry Algebras and their Representations

1.1 Supersymmetry Algebras

A big part of my research program can be explained just in terms of linear algebra and representation theory. The central concept here is called, by physicists, a "supersymmetry algebra", so to begin with let me say what I mean by this.

Supersymmetry algebras are examples of *super Lie algebras* defined over $\mathbb{R}$ or $\mathbb{C}$. A super Lie algebra is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a degree zero Lie bracket, satisfying graded versions of the axioms of a Lie algebra.

Remark 1.1. Super Lie algebras have the interesting feature that there can be non-zero elements Q such that $[Q, Q] = 0$. This means that differential graded Lie algebras appear almost from the beginning by treating the linear map $[Q, -]$ as a differential.

The Lie algebras we will discuss come in a few different flavors; I'll use the term *supersymmetry algebra* as a term that encompasses any and all of them.

Definition 1.2.

- A *supertranslation algebra* is a super Lie algebra whose even part is an abelian Lie algebra $\mathbb{R}^n$.
- A *super Poincaré algebra* is a super Lie algebra whose even part is the Lie algebra $\mathfrak{iso}(n)$ of infinitesimal isometries of $\mathbb{R}^n$ (or more generally $\mathbb{R}^{p,q}$, with indefinite signature).
- A *super conformal algebra* is a super Lie algebra whose even part contains the Lie algebra $\mathfrak{conf}(n)$ of infinitesimal conformal transformations of $\mathbb{R}^n$ (or more generally $\mathbb{R}^{p,q}$).

I've omitted some more technical assumptions here about the odd terms that are allowed. In each case there is a simple classification of all possible examples in each dimension n. For instance, if $n = 3$ or 4, the examples are indexed by a positive integer $\mathcal{N}$, so one talks about $\mathcal{N} = 1$ supersymmetry, $\mathcal{N} = 2$ supersymmetry and so on.

Supersymmetric field theories, at their heart, are built from representations of one of these super Lie algebras together with some extra structure, so we offer the following slogan:

Constructing supersymmetric theories involves studying nice classes of representations of supersymmetry algebras.

From this point of view twisting is a natural way of modifying a supersymmetric field theory. Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathcal{E})$ be a representation of a supersymmetry algebra $\mathfrak{g}$ and let Q be an odd element of $\mathfrak{g}$ such that $[Q, Q] = 0$. Then $\rho(Q)$ is a linear map $\mathcal{E} \to \mathcal{E}$ such that $\rho(Q)^2 = 0$.

Definition 1.3. The *twist* of $\mathcal{E}$ by Q is the cohomology with respect to $\rho(Q)$ [1]:

$$\mathcal{E}_Q = \ker(\rho(Q))/\mathrm{Im}(\rho(Q)).$$

So if we want to study the whole world of twists of supersymmetric field theories, we need to address the following overarching question.

Question 1.4.

1. For a given supersymmetry algebra $\mathfrak{g}$, what does the associated *nilpotence variety* look like? That is, what is
$$\mathcal{N}\mathrm{ilp}_{\mathfrak{g}} = \{Q \in \mathfrak{g}_1 \colon [Q, Q] = 0\}?$$
2. $\mathfrak{g}$ is the Lie algebra of a super Lie group G. What are the orbits of the even part G_0 on $\mathcal{N}\mathrm{ilp}_{\mathfrak{g}}$? Elements in the same orbit produce equivalent twists, so the space of possible twists is the orbit space [2] $\mathcal{N}\mathrm{ilp}_{\mathfrak{g}}/G$.

[1] Or we might remember the whole cochain complex $(\mathcal{E}, \rho(Q))$.
[2] When we do this carefully we use the language of stacks.

3. For any supersymmetric field theory $\mathcal{E}$, what do the twists $\mathcal{E}_Q$ look like as a family over $\mathcal{N}\mathrm{ilp}_{\mathfrak{g}}/G$?

Research Project 1.5. In [ES19] we answered questions 1 and 2 for all super Poincaré algebras (though over $\mathbb{C}$, not over $\mathbb{R}$) $\mathfrak{g}$ with $\dim(\mathfrak{g}_1) \leq 16$. Then in [ESW22] we answered question 3 for a very important class of supersymmetric field theories, the *super Yang–Mills theories*. These exist in all dimensions $n \leq 10$, so the classification involves a large number of possible cases.

Student Research Project 1.6. In dimension 3, there is a class of examples of supersymmetric field theory called *superconformal Chern–Simons theory*. They exist with $\mathcal{N} = 1, 2, \dots, 6$ and 8 [MFOME09]. An approachable project would involve computing the twists in some simple examples; the first non-trivial case shows up when $\mathcal{N} = 2$. Does the answer depend on the signature, i.e. for $\mathbb{R}^3$ versus $\mathbb{R}^{1,2}$? Understanding these theories is an active area of research among physicists – for instance they are important for the study of the M2 brane in string theory – but has not yet been tackled by mathematicians.

Research Project 1.7. In current joint work with Owen Gwilliam and Matteo Lotito [EGL23] we are studying the structure of twists of *superconformal* field theories in dimensions 3 and up. There are some concrete algebraic questions that we address, whose consequences I'll talk about a bit more in Example 2.7 in the next section. In brief we study the following:

- The superconformal nilpotence variety, and its orbits for the even part of the superconformal group.
- For each orbit $[Q]$, we compute the kernel $\mathfrak{z}_Q$ and the image $\mathfrak{b}_Q$ of the operator $[Q, -]$ as subalgebras of the even part of the superconformal algebra.

If we can put a Q-twisted superconformal field theory on an affine space where $\mathfrak{b}_Q$ includes the Lie algebra of isometries then the restriction of the Q-twisted theory to this affine space has very nice properties, as I will briefly discuss in the next section.

Student Research Project 1.8. In the summer of 2023 I ran a research project through the SURF (Science Undergraduate Research Fellowship) program at Amherst College – an eight week program intended to give students their first exposure to research in the sciences – about exactly this question. My students Osha Jones and Ziji Zhou investigated superconformal Lie algebras in dimension 3 (these are modelled by "orthosymplectic" super Lie algebras $\mathfrak{osp}(k|4)$) and gave answers to the two classification questions above over $\mathbb{C}$. This sort of project could be extended in many directions by other interested students, for instance seeing what happens over $\mathbb{R}$, and investigating the parallel questions for other simple super Lie algebras.

Research Project 1.9. In one final project, in current work with Brian Williams we are studying supersymmetric theories of gravity (or *supergravity*). The twists of supergravity play a central role in an exciting area of current research – the theory of *twisted holography* [CG18a] – but so far all the descriptions of twisted supergravity remain conjectural. This is because the descriptions of supergravity theory given in physics have not been formulated in a mathematically precise enough language to allow us to prove theorems about their twists. Williams and I have an approach by which we can write down supergravity theory in four dimensions (with the smallest non-trivial amount of symmetry that is possible, known as $\mathcal{N} = 1$ supersymmetry) using ideas from our earlier work [ESW22] as well as work of Baulieu, Bellon, Ouvry and Wallet [Bau+90]. Once we have done this we can use our previous work to compute the twist and compare it to the conjectural answer (which has to do with the Lie algebra of holomorphic vector fields on $\mathbb{C}^2$) .

Student Research Project 1.10. For further evidence of this conjecture in a more general context, that of a complex surface, geometrically inclined students could start with some specific complex surfaces, such as $\mathbb{CP}^2$ or Hirzebruch surfaces, and investigate the problem of lifting the solutions to the equations of motion in the conjectural twisted theory to solutions to the equations of motion in supergravity theory (which involve deformations of the metric on these surfaces).

1.2 Constructing Supersymmetric Theories

Another branch of my research involves the *construction* of supersymmetric field theories. One tool that I use to address these questions goes by the name "the pure spinor formalism", going back to work in the physics literature of Berkovits and Cederwall (e.g. [Ber01; Ber02; Ced18; Ced10]).

In order to describe this in a little more detail, let's consider a physically relevant structure to impose on a $\mathfrak{g}$-representation, a structure that is meaningful for modelling the classical fields of a field theory.

Definition 1.11. Let $\mathfrak{g}$ be a supersymmetry algebra in dimension n. A $\mathfrak{g}$-*multiplet* is a graded vector bundle E on $\mathbb{R}^n$ with a degree 1 differential d and an action ρ of $\mathfrak{g}$, so that:

- Both d and $\rho(X)$ for each $X \in \mathfrak{g}$ act by differential operators on the sections of E.
- The even part of $\mathfrak{g}$ acts "geometrically", meaning that it acts by the Lie derivative.
- For each $X \in \mathfrak{g}$, $\mathrm{d}\rho(X) + \rho(X)\mathrm{d} = 0$.

More generally we typically relax this definition, and only require ρ to be an action "up to homotopy" (in technical language, we ask for ρ to be an L_∞-action).

Thus a multiplet is a $\mathfrak{g}$-representation with additional "geometric" structure: it lives in the world of dg-vector bundles on $\mathbb{R}^n$. The pure spinor formalism had been thought of as a recipe for constructing multiplets, but in joint work with Hahner and Saberi we showed that, if one is willing to work in the right context, there is actually not just a construction, but an equivalence of categories! The other side of the correspondence is given as the category of (differential graded) modules over a (differential graded) commutative algebra that is well-known in representation theory.

Definition 1.12. If $\mathfrak{g}$ is a Lie algebra (or super Lie algebra), the *Chevalley–Eilenberg complex* $C^\bullet(\mathfrak{g})$ is the dg commutative algebra of the form

$$(\mathrm{Sym}^\bullet(\mathfrak{g}^*[-1]), \mathrm{d}_{\mathrm{CE}}),$$

where $\mathfrak{g}^*[-1]$ means we place $\mathfrak{g}^*$ in cohomological degree 1, and where d_{CE} is generated as a derivation by the linear map $\mathfrak{g}^* \to (\mathfrak{g}^*)^{\otimes 2}$ dual to the Lie bracket. The *cohomology* of $\mathfrak{g}$ is the cohomology of this complex.

Theorem 1.13 ([EHS23])**.** Let $\mathfrak{g}$ be a super Poincaré algebra in dimension n, and let $\mathfrak{t} \subseteq \mathfrak{g}$ be the corresponding supertranslation algebra. There is an equivalence of categories between the category of finitely generated $\mathfrak{g}$-multiplets and the category of finitely generated $C^\bullet(\mathfrak{t})$-modules that are $\mathfrak{so}(n)$ invariant.

Remark 1.14. If we study the spectrum $\mathrm{Spec}(C^\bullet(\mathfrak{t}))$, there is a map from the nilpotence variety

$$\mathcal{N}\mathrm{ilp}_{\mathfrak{g}} \to \mathrm{Spec}(C^\bullet(\mathfrak{t}))$$

that becomes an isomorphism on the reduced part of these schemes. As such we can think of $\mathrm{Spec}(C^\bullet(\mathfrak{t}))$ as an "enhancement" of the nilpotence variety by some "derived fuzz". Our theorem says that multiplets are equivalent to coherent sheaves on this enhanced nilpotence variety.

Student Research Project 1.15. Surprisingly, although the cohomology of supertranslation algebras $C^\bullet(\mathfrak{t})$ can be defined very concretely it doesn't seem to be known in many examples. An exciting project for a computationally inclined student would be to develop code for computing these cohomology groups. This would allow us to investigate important questions for the pure spinor formalism. For example, do these cohomology rings have duality pairings? In dimensions > 11 does the cohomology become trivial? We expect that it would, indeed this would follow from a conjecture of Hartshorne [Har74], but we are lacking empirical evidence.

Research Project 1.16. The really interesting thing to do with pure spinor theory is to try to build "interacting" supersymmetric field theories. These are multiplets with an extra piece of algebraic structure (very roughly speaking we want the data of a Lie bracket and a pairing on $\mathcal{E}$). There are various "folk results" in physics stating that interesting interacting supersymmetric theories only exist in certain dimensions (up to dimension 11) and for certain super Lie algebras (the odd part should have dimension at most 32).

In order to give a mathematical basis to these folk results we would like to translate these algebraic structures to the other side of the equivalence. The main step here is to understand what the tensor product of multiplets corresponds to on the $C^\bullet(\mathfrak{t})$-module side (it doesn't match the usual tensor product there). I'm working on performing this translation with Hahner, and we can see that it would lead to all sorts of exciting applications, including providing a proof of a conjecture of Costello and Li regarding what the twists of supergravity theories look like.

2 Topology – Factorization Algebras, TQFTs and Anomalies

2.1 Algebras of Observables

Quantum field theories have plenty of structure beyond that of a vector space, and in the case of twisted theories specifically, this extra structure interacts with topology in a fundamental way. Models for quantum field theory that appear centrally in my research include various *algebras of operators*, describing local observable quantities in a quantum field theory. These come in a few flavors.

- The most general model is that of a *factorization algebra*, as studied in [CG16; CG18b]. In brief, a factorization algebra on a manifold X assigns a vector space $\mathcal{F}(U)$ to each open set $U \subseteq X$, together with an *operator product*

  $$m^V_{U_1,U_2} \colon \mathcal{F}(U_1) \otimes \mathcal{F}(U_2) \to \mathcal{F}(V)$$

  for each pair U_1, U_2 of disjoint open subsets of an open set V. These products have to satisfy suitable coherence conditions when they are composed.

  Research Project 2.1. There is a general recipe that starts from the physicists' data defining a classical field theory – a collection of fields and an action functional – and produces a factorization algebra of quantum observables. In work with Gwilliam and Williams [EGW21] we constructed the factorization algebras associated to a class of field theories called *Kapustin–Witten theories*; these are examples of twists of supersymmetric field theories as described in the previous section. I studied these examples quite extensively in earlier work with Yoo [EY18; EY19; EY20].

  In this work we actually obtained these theories in families over an interesting parameter space whose points describe a choice of twist and a choice of "vacuum state" around which we study perturbations. Working over a choice of vacuum state is an example of the Higgs mechanism; Gwilliam and I explained how to interpret this mechanism using the modern language of derived algebraic geometry in an earlier work [EG21]

- There is a special class of factorization algebras that has been very widely studied in topology, the class of $\mathbb{E}_n$-algebras (or n-disk algebras). In homotopy theory the $\mathbb{E}_n$-algebras interpolate between associative algebras (which are roughly the same as $\mathbb{E}_1$ algebras) and commutative algebras (which are roughly the same as the $n \to \infty$ limit, the $\mathbb{E}_\infty$-algebras). This means that in an $\mathbb{E}_n$-algebra there is a multiplication associated to every embedding of two small n-disks in a bigger n-disk, but – crucially – every homotopy between two such configurations gives a homotopy between the multiplication operations.

  Remark 2.2. When we talk about "homotopy equivalence" here we could mean that we're studying algebras in topological spaces. More often I will still be thinking about linear algebra, but in a differential graded context. In this context homotopy equivalence means quasi-isomorphism of cochain complexes.

  In terms of quantum field theory, the quantum observables have the structure of an $\mathbb{E}_n$ algebra whenever the theory is "topological", meaning that it only depends on the topology of spacetime up to homotopy. In terms of twisted supersymmetric theories a result of mine with Safronov makes this idea precise.

  Theorem 2.3 ([ES19])**.** Let $\mathcal{E}$ be a supersymmetric theory on $\mathbb{R}^n$ and let Q be a square-zero supersymmetry with the property that every translation is in the image of $[Q,-]$ (this is called a topological supercharge). The algebra of quantum observables in the twisted theory $\mathcal{E}_Q$ canonically has the structure of an $\mathbb{E}_n$-algebra as long as it is rescaling invariant (the natural map from observables on a small disk to observables on a bigger concentric disk is a homotopy equivalence).

  Research Project 2.4. We can actually go a little further. If the twisted theory $\mathcal{E}_Q$ has an action of the group $\mathrm{SO}(n)$ of *rotations* in an appropriately "topological" sense, then the observables can be given the structure of what's known as a *framed* $\mathbb{E}_n$-algebra. Having a framed $\mathbb{E}_n$ algebra is particularly nice because we can then go from the observables on $\mathbb{R}^n$ to compute the space of observables on *any* oriented n-manifold M^n. This process is called *factorization homology*, as developed by Ayala and Francis [AF15].

  Research Project 2.5. Perhaps even more interestingly, sometimes one does not have an action of $\mathrm{SO}(n)$ itself on the quantum observables, but only of some subgroup $H \subseteq \mathrm{SO}(n)$ (or maybe $H \subseteq \mathrm{Spin}(n)$). The upshot to this story is that rather than being able to define an invariant of oriented n-manifolds, taking factorization homology gives a structure associated to manifolds with H-holonomy. In my work [ESW22] classifying the twists of super Yang–Mills theories we naturally come across all sorts of interesting examples. Often we only have an action of $\mathrm{U}(m)$ or $\mathrm{SU}(m)$, in which case we can compute factorization homology on complex or Calabi–Yau manifolds. But there are also more exotic examples in dimensions 7 and 8, giving invariants for manifolds with G_2 or $\mathrm{Spin}(7)$ special holonomy. While these theories have been studied by physicists [AOS97] they have not really appeared in the mathematics literature. By analyzing these theories we could obtain novel algebraic invariants for manifolds with special holonomy. This ties in with interesting student projects, which I describe in 2.10 below.

- One final context that we could work in is the world of *vertex algebras*. These are used to describe the observables of quantum field theories on $\mathbb{C}$ with conformal symmetry, though they have an independent life in the field of representation theory.

Research Project 2.6. In the case of Kapustin–Witten theories described in Example 2.1, in [EGW21] we obtained families of framed $\mathbb{E}_4$-algebras from the topological twists. There are also examples of points in this family that produce vertex algebras (the so-called "Kapustin-twists" [Kap06], they live on $\mathbb{R}^2 \times \mathbb{C}$ with an $\mathbb{E}_2$ product in the $\mathbb{R}^2$ direction and a vertex algebra structure in the $\mathbb{C}$ direction). In a follow-up to this work we plan to describe the vertex algebras that appear in this family, it would be very interesting to see how they relate to the constructions of vertex algebras that already appear in the literature.

Research Project 2.7. In my current work [EGL23] on twists of superconformal theories we give a general recipe for producing various $\mathbb{E}_n$, framed $\mathbb{E}_n$ and vertex algebras from a superconformal quantum field theory and a square zero odd element Q of the superconformal Lie algebra. As I discussed in Example 1.7 above, we can produce these interesting structures by looking for affine subspaces of the conformal compactification of $\mathbb{R}^{p,q}$ so that the symmetries of the subspace are Q-exact.

This construction provides a systematic generalization of an influential construction of vertex algebras from 4d superconformal field theories first described by Beem et al [Bee+15].

2.2 Anomalies

I should say something about what we actually have to *do* to obtain these nicely structured algebras of observables. The main computational input is – again – coming from linear algebra.

For example, let's focus on the case of $\mathbb{E}_n$ or framed $\mathbb{E}_n$ algebras. In this setting my theorem with Safronov guarantees their existence, and gives us an effective construction for them, provided that the translation and/or rotation action on the quantum observables is well-defined. Defining this action on the classical observables is very easy, but not all symmetries of classical observables continue to be well-defined at the quantum level.

So when can we extend from classical to quantum? Checking that this is possible involves evaluating certain cohomology classes called *anomalies*, and showing that they vanish. We can often study the cohomology groups where these anomalies live concretely using Lie algebra cohomology. Computing the classes themselves is done by evaluating Feynman diagrams.

Research Project 2.8. In work with Gwilliam [EG22] I showed that the relevant anomalies for obtaining a framed $\mathbb{E}_n$ algebra ("framing anomalies") are forced to vanish for a large class of topological quantum field theories, including many that arise from twisting.

Student Research Project 2.9. In the above work we computed the cohomology groups where the anomalies live, and found a class of examples where they vanish, but there is also a class of examples where the cohomology group does *not* vanish. In these cases it is important to actually compute the anomaly itself and see which cocycle we get. This computation is given by evaluating the weights of Feynman diagrams. Computing the weight of a diagram has two parts. First an algebraic part that is usually easy to calculate, and second an analytic part that involves evaluating a particular multivariate integral. These integrals are non-trivial, but they have certain tricks to them – we wrote some of these tricks down in the appendices to our paper [EWY18] – that an interested undergraduate student could learn. One example is a topological gauge theory in 6-dimensions called "6d BF theory". I suspect that the anomaly is typically non-zero here.

Student Research Project 2.10. In Project 2.5 I mentioned the existence of a 7-dimensional theory with an action of the group G_2. An exciting project would be to compute the G_2 framing anomaly for this theory. If this vanishes then the theory makes sense on any 7-manifold with a G_2 structure. This would involve learning about the group G_2, which is a very fun construction involving the octonions.

Research Project 2.11. In a paper with Williams [EW21] I introduced a class of field theories that make sense on Poisson manifolds called *Poisson BF Theories*. To quantize such a theory there is, again, an anomaly. For example, one could consider Poisson BF theories on $\mathbb{C}^{2n} \times \mathbb{R}^m$ where we have a Poisson structure that is holomorphic on the first factor and degenerate on the second factor. We showed that the anomaly vanishes when m is even or $m < 7$, but we don't know what happens in general. This ties in to the broader theory of Poisson manifolds, and the further interesting question of what happens in the curved case.

Student Research Project 2.12. In 2021 I mentored Jiaxi Tian for an REU at UMass Amherst about these anomalies. Jiaxi learned about Lie algebra cohomology and wrote some code to evaluate the relevant cohomology groups for anomalies of Poisson BF theories in low dimensions.

As a future project, we conjecture that the anomaly sometimes *does* vanish, and I have a counter-example in mind where the relevant cohomology group is interesting. Specifically, let us consider $\mathbb{C}^4 \times \mathbb{R}^9$, where the appropriate cohomology group is two-dimensional. The project I have in mind would involve setting up and computing a certain class in this cohomology and checking that we obtain a non-trivial cocycle, proving that these theories have the potential to be anomalous.

2.3 Defects

In this final section I will discuss research problems related to the notion of a *defect* for a quantum field theory. Defects describe situations where we have a manifold M and a submanifold D on which the field theory exhibits some sort of discontinuous behaviour. For example, there might be additional degrees of freedom along the submanifold D, or the fields might have a particular type of singularity along D. One might, for instance, think of $\mathbb{R} \subseteq \mathbb{R}^3$ thought of as the location a wire, and study charged particles constrained to move along the wire, but in the presence of an external electromagnetic field in $\mathbb{R}^3$.

The theory of defects can be investigated using the language of factorization algebras using an approach I have developed with Ivan Contreras and Owen Gwilliam [CEG23]. Our main new input is the following idea. On a manifold with boundary, we can make sense of the notion of a "quantum field theory with a boundary condition": there will be observables associated to two sorts of open subset, those disjoint from the boundary ("bulk observables"), and those that intersect the boundary ("boundary observables"). The technology for making these constructions go through was recently developed by Rabinovich [Rab21], and we can now take advantage of these developments.

Definition 2.13. A *defect* for a classical field theory $\mathcal{E}$ on a manifold M, along a submanifold $D \subseteq M$, is a choice of tubular neighborhood U for D together with a boundary condition for $\mathcal{E}$ along ∂U in the sense of Rabinovich.

In the setting where the theory $\mathcal{E}$ is topological in the directions normal to D, this notion does not depend on the choice of tubular neighborhood. In contrast, in the non-topological case a more refined notion would involve an equivalence class of boundary conditions defined for increasingly small neighborhoods of D.

There are a few examples that I am currently investigating. The first is foundational, but will lead to a fundamental example in the theory of vertex algebras

Research Project 2.14. We can describe the quantum field theory of a free boson on $\mathbb{C}$ – a very basic and fundamental example – in the presence of a finite set of point defects, labelled by integers. The objective is to realize this quantization as a sheaf of vertex algebras over the labelled configuration space of points in $\mathbb{C}$. I can then study the relationship between the cohomology of this sheaf and the lattice vertex algebra.

I am currently studying a second example in joint work with Contreras.

Research Project 2.15. If (M, π) is a Poisson manifold with an action of a group G, we can study a 3d gauge theory on $\mathbb{R}^3$ (the theory called "BF theory with gauge group G") in the presence of a defect along a choice of 2d submanifold

$\Sigma \subseteq \mathbb{R}^3$. The defect involves M, it is the "Poisson sigma model") with target M. We will compute the anomaly for this theory, and when it vanishes we can compute the algebra of observables along Σ. By choosing different points to perform perturbation theory around, we will end up being able to generalize existing calculations involving the Poisson sigma model, for instance of the partition function, to make sense in families over the moduli space of flat G-bundles on Σ.

Research Project 2.16. Another exciting example to investigate is that of Yang–Mills theory on $\mathbb{R}^4$, together with line defects along a set of parallel lines. The local quantum observables of this theory will produce a sheaf of factorization algebras over a labelled configuration space of points in $\mathbb{R}^3$ (the points where the parallel lines pass through an orthogonal hyperplane). This situation was studied by [AST13]. Specifically, the labels are given by elements of $Z(G) \times Z(G^\vee)$ where G is the gauge group of the theory and $G^\vee$ is a related group called the Langlands dual group. In particular, even though the perturbative theory is only sensitive to the Lie *algebra* of the gauge group, the family of theories over the configuration space depends on the specific form of the *group*.

References

[AF15] David Ayala and John Francis. "Factorization homology of topological manifolds". *J. Topol.* 8.4 (2015), pp. 1045–1084. URL: `https://doi.org/10.1112/jtopol/jtv028`.

[AOS97] Bobby Samir Acharya, M. O'Loughlin, and Bill J. Spence. "Higher dimensional analogs of Donaldson-Witten theory". *Nucl. Phys. B* 503 (1997), pp. 657–674. arXiv: `hep-th/9705138`.

[AST13] Ofer Aharony, Nathan Seiberg, and Yuji Tachikawa. "Reading between the lines of four-dimensional gauge theories". *JHEP* 08 (2013), p. 115. arXiv: `1305.0318 [hep-th]`.

[Bau+90] Laurent Baulieu, Marc P. Bellon, Stephane Ouvry, and Jean-Christophe Wallet. "Balatin-Vilkovisky analysis of supersymmetric systems". *Phys. Lett. B* 252 (1990), pp. 387–394.

[Bee+15] Christopher Beem et al. "Infinite Chiral Symmetry in Four Dimensions". *Communications in Mathematical Physics* 336.3 (2015), pp. 1359–1433. URL: `https://doi.org/10.1007%2Fs00220-014-2272-x`.

[Ber01] Nathan Berkovits. "Covariant quantization of the superparticle using pure spinors". *JHEP* 09 (2001), p. 016. arXiv: `hep-th/0105050`.

[Ber02] Nathan Berkovits. "Towards covariant quantization of the supermembrane". *JHEP* 09 (2002), p. 051. arXiv: `hep-th/0201151`.

[Ced10] Martin Cederwall. "Towards a manifestly supersymmetric action for 11-dimensional supergravity". *JHEP* 01 (2010), p. 117. arXiv: `0912.1814 [hep-th]`.

[Ced18] Martin Cederwall. "Pure spinor superspace action for D = 6, N = 1 super-Yang-Mills theory". *JHEP* 05 (2018), p. 115. arXiv: `1712.02284 [hep-th]`.

[CEG23] Ivan Contreras, Chris Elliott, and Owen Gwilliam. "Defects via factorization algebras". *Lett. Math. Phys.* 113.2 (2023), Paper No. 46, 26. URL: `https://doi.org/10.1007/s11005-023-01670-2`.

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory, Volume I*. New Mathematical Monographs 31. Cambridge University Press, 2016.

[CG18a] Kevin Costello and Davide Gaiotto. "Twisted holography" (2018). arXiv: `1812.09257`.

[CG18b] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 2*. 2018. URL: `https://people.math.umass.edu/~gwilliam/vol2may8.pdf`.

[EG21] Chris Elliott and Owen Gwilliam. "Spontaneous symmetry breaking: a view from derived geometry". *J. Geom. Phys.* 162 (2021), Paper No. 104096, 15. URL: `https://doi.org/10.1016/j.geomphys.2020.104096`.

[EG22] Chris Elliott and Owen Gwilliam. "Framed $\mathbb{E}_n$-Algebras from Quantum Field Theory". *arXiv preprint arXiv:2204.03702* (2022).

[EGL23] Chris Elliott, Owen Gwilliam, and Matteo Lotito. "Twists of Superconformal Algebras". *Preliminary version available at https://chrisjelliott.github.io/Twists_of_Superconformal_Algebras.pdf* (2023).

[EGW21] Chris Elliott, Owen Gwilliam, and Brian R Williams. "Higher Deformation Quantization for Kapustin–Witten Theories". *arXiv preprint arXiv:2108.13392* (2021).

[EHS23] Chris Elliott, Fabian Hahner, and Ingmar Saberi. “The derived pure spinor formalism as an equivalence of categories”. *SIGMA Symmetry Integrability Geom. Methods Appl.* 19 (2023), Paper No. 022, 37. URL: https://doi.org/10.3842/SIGMA.2023.022.

[EP19] Chris Elliott and Vasily Pestun. “Multiplicative Hitchin systems and supersymmetric gauge theory”. *Selecta Math. (N.S.)* 25.4 (2019), Paper No. 64, 82. URL: https://doi.org/10.1007/s00029-019-0510-y.

[ES19] Chris Elliott and Pavel Safronov. “Topological twists of supersymmetric algebras of observables”. *Comm. Math. Phys.* 371.2 (2019), pp. 727–786. arXiv: 1805.10806 [math-ph].

[ESW22] Chris Elliott, Pavel Safronov, and Brian R. Williams. “A taxonomy of twists of supersymmetric Yang–Mills theory”. *Selecta Math. (N.S.)* 28.4 (2022), Paper No. 73. URL: https://doi.org/10.1007/s00029-022-00786-y.

[EW21] Chris Elliott and Brian R. Williams. “Holomorphic Poisson field theories”. *High. Struct.* 5.1 (2021), pp. 282–309.

[EWY18] Chris Elliott, Brian R Williams, and Philsang Yoo. “Asymptotic freedom in the BV formalism”. *J. Geom. Phys.* 123 (2018), pp. 246–283. arXiv: 1702.05973 [math-ph].

[EY18] Chris Elliott and Philsang Yoo. “Geometric Langlands Twists of $N = 4$ Gauge Theory from Derived Algebraic Geometry”. *Advances in Theoretical and Mathematical Physics* 22.3 (2018), pp. 615–708.

[EY19] Chris Elliott and Philsang Yoo. “A Physical Origin for Singular Support Conditions in Geometric Langlands”. *Communications in Mathematical Physics* 368 (3 2019), pp. 985–1050.

[EY20] Chris Elliott and Philsang Yoo. “Quantum Geometric Langlands Categories from $\mathcal{N} = 4$ Super Yang–Mills Theory” (2020). arXiv: 2008.10988.

[Har74] Robin Hartshorne. “Varieties of small codimension in projective space”. *Bull. Amer. Math. Soc.* 80 (1974), pp. 1017–1032. URL: https://doi.org/10.1090/S0002-9904-1974-13612-8.

[Kap06] Anton Kapustin. “Holomorphic reduction of $N = 2$ gauge theories, Wilson-'t Hooft operators, and S-duality”. *arXiv preprint hep-th/0612119* (2006).

[MFOME09] Paul de Medeiros, José Figueroa-O'Farrill, and Elena Méndez-Escobar. “Superpotentials for superconformal Chern-Simons theories from representation theory”. *J. Phys. A* 42.48 (2009), pp. 485204, 56. URL: https://doi.org/10.1088/1751-8113/42/48/485204.

[Rab21] Eugene Rabinovich. “Factorization Algebras for Bulk-Boundary Systems”. arXiv preprint arXiv:2111.01757. PhD thesis. University of California, Berkeley, 2021.