Geometric Langlands for Physicists / Kapustin–Witten for Mathematicians

Lecture 1

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Plan for Today

- Introduce the Hitchin system, and talk about some of its structure.
- Discuss what it means to quantize the Hitchin system.
- Talk about how to describe this quantization; this will lead us to a version of the geometric Langlands conjecture.

Next time we'll talk about how all the objects in todays lecture arise in gauge theory.

Higgs Bundles

Let C be a Riemann surface, and let G be a complex Lie group (e.g. $G = \mathrm{GL}(n; \mathbb{C})$).

Definition

A Higgs bundle on C for the group G is a pair (P, ϕ) where:

- P is a holomorphic G-bundle on C (e.g. for $\mathrm{GL}(n; \mathbb{C})$ think of a holomorphic vector bundle E).
- ϕ is a holomorphic 1-form taking values in $\mathfrak{g}_P$ (e.g. for $\mathrm{GL}(n; \mathbb{C})$ think of an endomorphism of E i.e. a matrix-valued section).

There is a moduli space of Higgs bundles on C that we denote as

$$\mathrm{Higgs}_G(C).$$

The name Higgs bundle comes by analogy to Yang–Mills–Higgs theory.

- Holomorphic structure on $P \approx$ certain connections A on P solving the Yang–Mills equations (in 4d: instantons).
- Holomorphic 1-form $\phi \approx \mathrm{d}_A$-closed 1-form ϕ.

The space of Higgs bundles is a holomorphic version of the space of minimum energy solutions to the Yang–Mills–Higgs equations (in 2d, but a similar definition can be made in other dimensions).

The space $\mathrm{Higgs}_G(C)$ has a nice description as follows. There's a forgetful map down to the moduli space of holomorphic G-bundles (forget ϕ)

$$\pi\colon \mathrm{Higgs}_G(C) \to \mathrm{Bun}_G(C).$$

It turns out that this map is identified with the projection from the cotangent bundle of $\mathrm{Bun}_G(C)$

$$\mathrm{Higgs}_G(C) \cong T^* \mathrm{Bun}_G(C).$$

Using this fact, we can identify a few things straight away:

- $\dim_{\mathbb{C}}(\mathrm{Higgs}_G(C)) = 2\dim_{\mathbb{C}}(\mathrm{Bun}_G(C)) = (2g-2)\dim(G)$ where g is the genus of C (if $g > 1$).
- $\mathrm{Higgs}_G(C)$ has a holomorphic symplectic structure.

Structures on $\mathrm{Higgs}_G(C)$

Actually the moduli space has even richer structure! Two different closely related stories.

1. Because of the symplectic structure, it makes sense to talk about the Poisson bracket of two holomorphic functions on $\mathrm{Higgs}_G(C)$. It turns out that we can find a whole collection

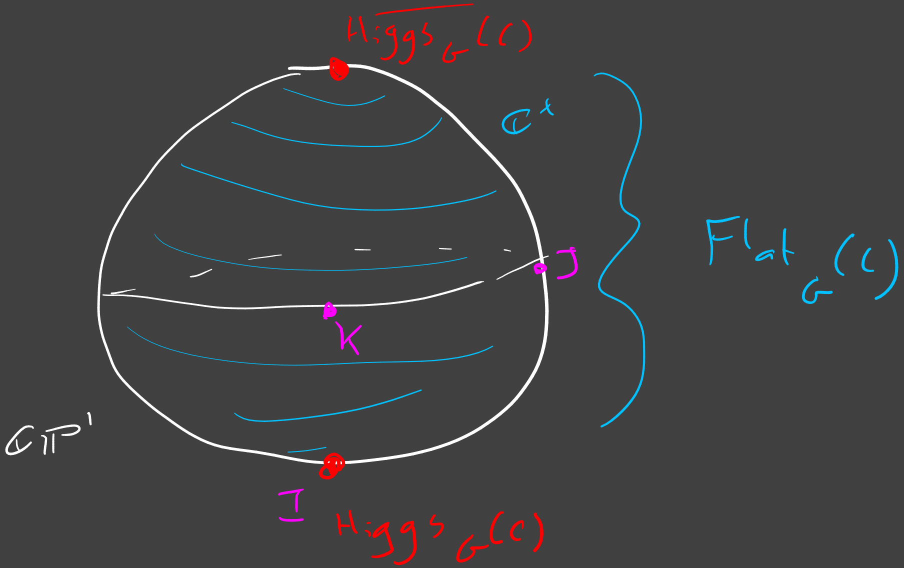
$$h_1, \dots, h_d$$

of holomorphic functions where $d = (g-1)\dim(G)$ is half the dimension of $\mathrm{Higgs}_G(C)$ with the property that $\{h_i, h_j\} = 0$ for any i and j. This is the structure of an integrable system. Geometric point of view: this is the same as a holomorphic map called the Hitchin fibration

$$p\colon \mathrm{Higgs}_G(C) \to \mathbb{C}^d$$

with some nice properties. You can describe p concretely in terms of the characteristic polynomial of the Higgs field.

2. There is more than one complex structure on $\mathrm{Higgs}_G(C)$, in fact there is a whole $\mathbb{CP}^1$ family of such structures! $\mathrm{Higgs}_G(C)$ is an example of a hyperkähler space.



Quantization

We will study the canonical quantization of $\mathsf{Higgs}_G(C)$. Using the identification with a cotangent bundle $T^* \mathsf{Bun}_G(C)$, we can quantize the ring of functions on $\mathsf{Higgs}_G(C)$:

$$\mathcal{O}(\mathsf{Higgs}_G(C)) \rightsquigarrow \mathrm{DiffOp}(\mathsf{Bun}_G(C)).$$

Question (Quantization of the Hitchin system)

Can we find commuting differential operators $H_1, \ldots H_d$ that quantize the classical Hamiltonians $h_1, \ldots h_d$?

Answer (Beilinson–Drinfeld 1991): Yes, and they can be described in terms of the classical Hitchin system of a different "dual" group!

Question (Spectral problem)

Describe the simultaneous solutions to the equation

$$H_j(\phi) = a_j(\phi),$$

for $a \in \mathbb{C}^d$.

Fancy terminology: think of the space of solutions as the "D-module"

$$M_a = \mathrm{DiffOp}(\mathrm{Bun}_G(C))/(H_j - a_j).$$

The Langlands dual group

To the group G we can cook up its Langlands dual group $G^\vee$ combinatorially. "Reverse the root data": roots/characters of G are coroots/cocharacters of $G^\vee$ and vice versa. In some examples:

$$\begin{aligned} \mathrm{GL}(n;\mathbb{C}) &\leftrightsquigarrow \mathrm{GL}(n;\mathbb{C}) \\ \mathrm{SL}(n;\mathbb{C}) &\leftrightsquigarrow \mathrm{PGL}(n;\mathbb{C}) \\ \mathrm{SO}(2n+1;\mathbb{C}) &\leftrightsquigarrow \mathrm{Sp}(2n;\mathbb{C}). \end{aligned}$$

Suggestively, this duality operation also shows up in electric-magnetic duality.

Geometric Langlands duality: first picture

The solutions to the two quantization problems take the following flavour.

$$\text{Quantization of } \mathsf{Higgs}_G(C) \leftrightarrow \text{Deformation of } \mathsf{Higgs}_{G^\vee}(C).$$

Here deformation means deformation of the complex structure: going from $\mathsf{Higgs}_{G^\vee}(C)$ to $\mathrm{Flat}_{G^\vee}(C)$.

For example, the D-modules generated by $H_j - a_j$ can be associated to points in $\mathrm{Flat}_{G^\vee}(C)$. I'll say more about this in a second, but first let me state a strong (conjectural) version of this duality.

Conjecture (Beilinson–Drinfeld “best hope”)

There is an equivalence of categories

$$\{\textit{D-modules on } \mathsf{Bun}_G(C)\} \leftrightarrow \{\textit{Coherent sheaves on } \mathrm{Flat}_{G^\vee}(C)\}.$$

Examples of coherent sheaves are given by choosing a subspace $Y \subseteq \mathrm{Flat}_{G^\vee}(C)$ and specifying a vector bundle on Y. The simplest case of this is when Y is just a point!

Hecke Eigensheaves

Rough idea: there are nice operations one can perform on D-modules on $\mathrm{Bun}_G(C)$ called Hecke transformations. They take place locally at a point in C. The operations commute with one another, so one can try to decompose D-modules into simultaneous eigenvectors: such things are called Hecke eigensheaves.

By eigenvector in this context, we mean $A_x(M) = V_x \otimes M$, where x is a point in C, and V_x is a vector space.

Theorem
The D-module M_a solving the spectral problem is a Hecke eigensheaf with eigenvalue $V(a)$ in $\mathrm{Flat}_{G^\vee}(C)$.

The geometric Langlands conjecture would say that all points in $\mathrm{Flat}_{G^\vee}(C)$ arise as eigenvalues for some Hecke eigensheaf.