Hecke modifications

Aron Heleodoro

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1 Introduction

The interest on Hecke modifications in the geometrical Langlands program comes as a natural categorification of the product in the spherical Hecke algebra, which one associates with an irreducible unramified representation of GL_n (or any algebraic group) on a local field, or the completion of a function field around a point for curves over a finite field. In the case over $\mathbb{C}$, not only one has a more natural way of connecting the local and global pictures but also certain questions of concern in the number field case simply do not arise.

Let X be a smooth projective curve over $\mathbb{C}$, and G a (semisimple) reductive connected algebraic group. One form of the Langlands geometric conjecture is that to which ${}^LG-$local system E over X[1], one can associate a Hecke eigensheaf $\mathscr{L}_E$ on the (algebraic) stack Bun_G of principal $G-$bundles on X. In particular for GL_n these are, respectively, vector bundles with a flat connection and (perverse) sheaves (or equivalently $D-$modules) on the moduli space of isomorphism classes of rank n vector bundles over X. Moreover, these D-modules satisfy the Hecke conditions. To formulate the Hecke conditions we need to introduce the so-called Hecke modifications, which our seminar hopes to relate to 't Hooft operators on quantum field theories.

[1] One can take as a definition of an ${}^LG$-local system on X a homomorphism $\rho : \pi_1(X, x_0) \to^L G(\mathbb{C})$.

2 Hecke modifications

Let $x \in X$, we consider the set of triples $\text{Hecke}_x = \{(\mathscr{M}, \mathscr{M}') \in \text{Bun}_G \times \text{Bun}_G,\ \beta : \mathscr{M}|_{X\setminus x} \to \mathscr{M}'|_{X\setminus x}\}$, that is two isomorphism classes of principal $G-$bundles over X, with a bundle isomorphism β away from the point x. We have natural projections $h^{\leftarrow}(\mathscr{M}, \mathscr{M}') = \mathscr{M}$ and $h^{\rightarrow}(\mathscr{M}, \mathscr{M}') = \mathscr{M}'$. Now given a sheaf $\mathscr{F}$ over the space Bun_G, we can define its Hecke modification at the point x as

$$\mathbb{H}_x(\mathscr{F}) = h^{\rightarrow}_* \left(h^{\leftarrow *}(\mathscr{F})\right).$$

This is a very abstract definition and one needs to know specifically which kind of sheaves one wants to consider and how to make sense of these and of sheaf operations over an algebraic stack. We will try to give a more intuitive picture of what is happening without entering in the technicalities of any of the aforementioned.

Morally by the *function-faisceaux* dictionary, one can think of a sheaf $\mathscr{F}$ as a function $f(x)$ on the space Bun_G. Hence the push-pull operation on $f(x)$ is like an integral transform, $h^{\leftarrow *}(f)(x, y)$ is f considered as a two variable function, with trivial dependence on y. And $h^{\rightarrow}_*$ integrates over the fibers of the projection on the right factor. Our kernel here should be a characteristic function on $\text{Bun}_G \times \text{Bun}_G$ which realize pairs of principal $G-$bundles over X which are isomorphic away from a fixed point x.

More abstractly, and this is where the geometric aspects give a more natural way of globalizing these constructions as opposed to the number theory case of the program, we can consider Hecke the data of

$$\text{Hecke} = \{(x, \mathscr{M}, \mathscr{M}') \in X \times \text{Bun}_G \times \text{Bun}_G,\ \beta : \mathscr{M}|_{X\setminus x} \to \mathscr{M}'|_{X\setminus x}\}.$$

And the projection $\text{supp} \times h^{\rightarrow} : \text{Hecke} \to X \times \text{Bun}_G$ which is just $(x, \mathscr{M}')$. Then form the global Hecke modification by

$$\mathbb{H}(\mathscr{F}) = R\left(\text{supp} \times h^{\rightarrow}\right)_* \left(h^{\leftarrow *}(\mathscr{F})\right),$$

where one actually has to take the derived functor of the push-forward. This gives an operation from (derived) sheaves on Bun_G, to (derived) sheaves on $X \times \text{Bun}_G$, and the Hecke condition with respect to a local system E can be heuristically written as[2]

$$\mathbb{H}(\mathscr{F}) = E \boxtimes \mathscr{F}.$$

[2]The formula we gave is not precise. Actually one needs a condition which is different on different fibers of the projection and only has a simple form for irreducible local systems

We will make it precise in special cases to follow.

Note that one has a certain asymmetry on this formulaion of the geometric Langlands conjecture, because we considered special kind of sheaves *on* X on the lefthand side, and associated to it a special kind of sheaf *over* Bun_G on the righthand side. We could furthermore consider a certain moduli stack on the lefthand side also, i.e. $\mathrm{Loc}_{^LG}$, the moduli stack of ^LG local systems on X, and consider a special kind of sheaves on it, and for each one of them associate a Hecke eigensheaf. However it is subject of current research which kind of sheaves on the lefthand side form the appropriate category to consider.

With that abstract construction in mind one will now specialize to GL_n and then to $n = 1$ to try to have a more intuitive understanding of what is happening.

3 $G = GL_n$

In this case, as we already mentioned, the Langlands correpondence associates to a rank n vector bundle with a flat connection on X a Hecke eigensheaf on the moduli space of rank n vector bundles over X. The Hecke modification can be described more geometrically. To do that we will introduce some notation. Say $\mathscr{M} \in \mathrm{Bun}_n$, where by Bun_n we understand the sheaf of locally free quasi-coherent sheaves of rank n. One wants to understand, how many $\tilde{\beta} : \mathscr{M} \to \mathscr{M}'$ map of sheaves exist, such that the restriction to a formal neighbourhood which does not contain x is an isomorphism. Clearly, they can only differ at the point x. Say we fix a local parameter t at x, then we can model the completion of the local ring at x as $\mathbb{C}[[t]]$. Since this is a map of sheaves the compatibility conditions allow us to assume without lost of generatlity, that $\mathscr{M}_x \cong \mathbb{C}((t))^n$, and that the stalk at $\mathscr{M}_x$ is a sub-$\mathbb{C}[[t]]$-module of $\mathscr{M}'_x$ of rank n. This does not leave many options, and we can separate them by asking the jump in degree to be locally i. In other words, the length of $\mathscr{M}'_x/\mathscr{M}_x$ to be $i \geq 0$[3].

This gives us a stratification of Hecke_x, which we write as $\mathrm{Hecke}_{x,i}$ and

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E. One also would like to respect the grading of the derived categories of sheaves, so one has to be careful with the degrees. Finally, since we want to preserve the categorical information of the construction, it is natural not to ask for equalities but isomorphism on the Hecke conditions.

[3] One can clearly see that $\mathscr{M}_x \otimes_{\mathbb{C}[[t]]} \mathbb{C}((t)) \cong \mathscr{M}'_x$, i.e. $\mathscr{M}|_{\mathbb{D}_x^\times} \cong \mathscr{M}'|_{\mathbb{D}_x^\times}$, where $\mathbb{D}_x^\times$ is the formal punctured disk around x.

that can be described as

$$\text{Hecke}_{x,i} = \{(\mathscr{M}, \mathscr{M}') \in \text{Bun}_n \times \text{Bun}_n,\ \beta : \mathscr{M} \hookrightarrow \mathscr{M}',\ \text{length}(\mathscr{M}'_x / \mathscr{M}_x) = i\}.$$

Note that it allows this quotient to be any sheaf of length i supported on x, not necessarily a direct sum of skyscraper sheaves. [One can verify that the fibers of this morphism, over any of the two factors are $Gr(i,n)$ the Grassmanian of subspaces of dimension i inside n.] We actually can consider the global modification restricted to the subspaces as defined above, we write $\mathbb{H}_i$ for this Hecke modification. It is a functor from the derived category of sheaves on Bun_n to the derived category of sheaves on $X \times \text{Bun}_n$. This allows us to be more precise about the Hecke eigenproperty. We say the local system E of rank n over X is an eigenvalue for the sheaf $\mathscr{F}$, if we have isomorphisms

$$\imath_i : \mathbb{H}_i(\mathscr{F}) \xrightarrow{\simeq} \wedge^i E \boxtimes \mathscr{F}[i(n-i)], \quad i = 1, \ldots, n.$$

We need to perform a shift on degree to have the box product with $\wedge^i E$ to land in the same degree as the Hecke modification. One can check that this indeed gives back the Hecke eigenvalue conditions for the classical Hecke algebras by considering the curve X over a finite field $\mathbb{F}_p$. Since this is not our main interest in this seminar I would just sketch this connection in a later section.

4 $G = GL_1$

There are two particular features of the GL_1 case which make it more concrete than the general case just discussed. One is a geometric realization of the moduli space Bun_G, the other is a more explicit description of the Hecke modification as the Fourier-Mukai transform.

The local systems one consider in this case are just line bundles with a flat connetion E over X. On the other side, the moduli space of line bundles over X forms an actual scheme $Pic(X)$, the Picard variety of X. The Hecke modification as defined before in this case gives a map,

$$\begin{aligned} h^{\rightarrow} : X \times Pic(X) &\longrightarrow Pic(X) \\ (x, \mathscr{L}) &\longmapsto \mathscr{L}(x). \end{aligned}$$

Note that the only now trivial isomorphism away from a point is the twisting of the line bundle at that point. So the Hecke modification is just $h^{\rightarrow *}(\mathscr{F})$, and the Hecke condition reduces to $h^{\rightarrow *}(\mathscr{F}) \cong E \boxtimes \mathscr{F}$.

Now I will briefly describe a direct construction of the Hecke eigensheaf from the local system E. We write $X^{(d)}$ for the d-th symmetric product of X. And consider the map $h^{\rightarrow} : X \times X^{(d)} \rightarrow X^{(d+1)}$, which to a point x and a degree d divisor D associates $D + x$. We want to construct a sheaf $Sym_d(E)$ on each $X^{(d)}$ which satisfy the analog to the Hecke condition

$$E \boxtimes Sym_d(E) \xrightarrow{\simeq} h^{\rightarrow *} Sym_{d+1}(E).$$

Let $s^d : X^d \rightarrow X^{(d)}$ be the symmetrization map. We define $Sym_d(E) = \left(s^d_* \left(E^{\boxtimes d}\right)\right)^{S_d}$. It is just a verification that this satisfies the condition above.

To define the sheaf on $Pic(X)$ we use the Abel-Jacobi map from $\pi_d : X^{(d)} \rightarrow Jac(X)_d$, the degree d part of the Jacobian. This map associate to d points of X the corresponding divisor, which can also be seem as a line bundle of degree d, hence an element of $Jac(X)$. For d sufficiently large the fibers of this map are projective spaces and moreover have the same dimension over each line bundle. In fact, if the degree of $\mathscr{L}$ is greater than $2g - 2$, by Riemann-Roch dim $H^1(X, \mathscr{L}) = 0$, which gives that the space of sections of $\mathscr{L}$ is non-empty. Now to ask for a divisor D, s.t. $\mathscr{O}(D) = L$ is equivalent to consider $\mathbb{P}H^0(X, \mathscr{L})$.

We claim the sheaf $Sym_d(E)$ descends to $Pic(X)_d$, for $d \geq 2g - 1$. Since the sheaf E is locally constant, and the projective spaces are simply-connected, hence E is constant on the fibers of π_d. We call $Aut_d(E)$ the sheaf on $Pic(X)_d$. We can write a commutative diagram between the actions of $h^{\rightarrow}$ on the symmetric power of X and on the $Pic(X)$ at the same degree, connected by the Abel-Jacobi map. The commutativity of the diagram gives the Hecke condition for the components of the Picard of degree $d \geq 2g - 1$.

Finally, to extend the sheaf $Aut_d(E)$ to a lower degree we observe that if $Aut_{d-1}(E)$ were defined it would have to satisfy the Hecke condition

$$E \boxtimes Aut_{d-1}(E) \cong h^{\rightarrow *} Aut_d(E).$$

Restricting to a point $x \in X$ we can write $E_x \otimes Aut_{d-1}(E) \cong h_x^{\rightarrow *}(Aut_d(E))$ [4]. Since the stalk is just a vector space, we get $Aut_{d-1}(E) \cong E_x^* \otimes Aut_d(E)$.

[4] Clarifying

$$\begin{aligned} h_x^{\rightarrow} : & Pic(X) \rightarrow Pic(X) \\ & \mathscr{L} \rightarrow \mathscr{L}. \end{aligned}$$

And the box product on the stalk is, by definition, just the tensor product.

One still have to check that this reverse induction does not depend on the point x, which we leave to the reader's amusement.

This proves the correspondence on the case of GL_1, and actually this argument holds for fields of positive characteristic. One remarkable thing, which is what makes the geometric version of the Langlands program interesting, is that the descent condition here is very simple, and is solved by the simply-connected fibers argument due to Deligne. Even on the abelian case for number fields the descent condition is a non-trivial reciprocity law in abelian class field theory. This gives an ideia of how the geometric version has an easier local to global passage.

We make some remarks as how a generalized Fourier-Mukai transform can be used to establish this correspondence. We denote by Loc_1 the space of degree 1 local system on X, i.e. holomorphic line bundles with a holomorphic (hence flat) connection. Let $(\mathscr{F}, \nabla) \in \mathrm{Loc}_1$, since $\mathscr{F}$ has a flat connection, its first Chern class vanishes, so $\deg(\mathscr{F}) = 0$. We can then map $p : \mathrm{Loc}_1 \to Jac$ by just forgetting the connection, where $Jac = Pic_0$, i.e. the degree 0 part of the Picard variety. We look at the fibers of p, since any holomorphic line bundle $\mathscr{F}$ on X has a holomorphic connection, we can fix a connection ∇, and the fiber of p is $\nabla' = \nabla + \omega$, where ω is a holomorphic 1-form. By Serre duality $H^0(X, \mathscr{O}) \cong H^1(X, \Omega)^*$, so the fibers of p are dual to $H^1(X, \omega)$ which are the fibers of the cotangent bundle over Jac.[5].

From a local system $\mathscr{F} \in \mathrm{Loc}_1$ on X, we can construct a holomorphic line bundle with a (flat) holomorphic connection on Jac, i.e. a rank 1 local system. It goes like this: the data of $\mathscr{F}$ is the same as a group homomorphism from $\pi_1(X) \to \mathbb{C}^\times$, since $\mathbb{C}^\times$ is abelian this map factors through $H_1(X, \mathbb{Z})$ (the commutator subgroup of $\pi_1(X)$). In a Riemann surface X, we can identify $H_1(X, \mathbb{Z})$ with $H^1(X, \mathbb{Z})$, and we know $Jac \cong H^1(X, \mathscr{O})/H^1(X, \mathbb{Z}) \cong \mathbb{C}^g/H^1(X, \mathbb{Z})$ so we get a map from $\pi_1(Jac) \to \mathbb{C}^\times$.

Now let's consider the product $\mathrm{Loc}_1 \times Jac$ and the (almost tautological) bundle $\mathscr{P}$ over it defined by the restriction to $(\mathscr{F}', \nabla') \times Jac$ is the line bundle with connection $(\mathscr{F}', \nabla')$ on Jac. If we consider the projections p_1, p_2 to Loc_1 and Jac respectively, we obtain functors F and G between the bounded derived category of coherent $\mathscr{O}-$modules on Loc_1 to the derived category of $D-$modules on Jac:

$$F : \mathscr{M} \mapsto Rp_{1*}p_2^*(\mathscr{M} \otimes \mathscr{P}) \quad G : \mathscr{N} \mapsto Rp_{2*}p_1^*(\mathscr{N} \otimes \mathscr{P}).$$

[5]Sometimes one says that Loc_1 is the twisted cotangent bundle to Jac

In particular, if $\mathscr{O}_E$ is the skyscraper sheaf at $E = (\mathscr{F}, \nabla) \in \mathrm{Loc}_1$, then $G(\mathscr{O}_E)$ is (the degree zero component of) Aut_E, the Hecke eigensheaf on Pic.

Rothstein and Laumon proved that these functors establish an equivalence between the two categories. Thus, we can think of the Hecke eigensheaves on Pic (and in general Bun_G) as a conterpart of the skyscraper sheaves on Loc_1, so as building blocks of the category of $D-$modules on Pic.

5 Comparison with curves over finite fields

The Langlands program establishes a connection between representations of the absolute Galois group of a number field and certain automorphic representations. In the Galois side, we have a natural action of Frobenius elements, on finite dimensional representation - the Langlands correspondence also say we should have some operation on the automorphic side to which this Frobenius action corresponds. This is the what the Hecke operators accomplish.

We will actually consider the construction over a function field, because this lends to a closer analogy to the geometric case over $\mathbb{C}$. Let F be the function field of a smooth projective curve X over the finite field $\mathbb{F}_p$. One can consider the adeles over this curve X, $\mathbb{A}_X = \prod_{x \in X} F_x$, where F_x is the fraction field of the completion the local ring at the point x. If one chooses a rational function t vanishing at x one can identify (though non-canonically) this local field with $\mathbb{F}_x((t))$, here $\mathbb{F}_x$ is the residue field at the point x, this can be any finite extension of the field $\mathbb{F}_p$. We will denote by $\mathscr{O}_x$ its ring of integers, which in this case is just the power series $\mathbb{F}_x[[t]]$. The side of the Langlands program we are interested in is concerned with automorphic representation of $GL_n(\mathbb{A})$.

Let $GL_n(F)$ be the algebraic group GL_n with base F and $K = \prod_{x \in X} GL_n(\mathscr{O}_x)$. K is the maximal compact subgroup of $GL_n(\mathbb{A})$. We consider the action of $GL_n(F)$ on $GL_n(\mathbb{A})$ diagonally, and form the quotient $GL_n(F)\backslash GL_n(\mathbb{A})$. Let $\chi : GL_n(\mathbb{A}) \to \mathbb{C}^\times$ be a Grossencharacter, i.e. a character which vanishes on a finite (index) subgroup. We will be interested on the space of locally constant functions on this quotient $\mathscr{C}_\chi(GL_n(F)\backslash GL_n(\mathbb{A}))$. The action of $GL_n(\mathbb{A})$ on this function space is given by

$$h \cdot f(g) = f(gh), \quad \text{for } h \in GL_n(\mathbb{A}) \text{ and } f \in \mathscr{C}_\chi.$$

With this set up, we can give the conditions asked for $\mathscr{C}_\chi$ to be an automorphic representation

(i) The orbit $K \cdot f$ is finite-dimensional.

(ii) The central elements of $GL_n(\mathbb{A})$ act by the character, i.e. $f(g\alpha) = \chi(\alpha)f(g)$, for $\alpha \in \mathbb{A}^\times$.

(iii) Cuspidality. [6]

The reason we consider this particular cuspidal automorphic representation is that every irreducible (cuspidal) automorphic representation appears in it with multiplicity one. This is a theorem of Piatetski-Shapiro.

Say π is an irreducible cuspidal automorphic representation so defined. We can decompose it as $\pi = \otimes'_{x \in X} \pi_x$. Here each π_x is an irreducible representation of $GL_n(F_x)$ and the restricted product means that all but finitely many of the π_x are *unramified* representations. What this means in this case is that there exists a nonzero element $v_x \in \pi_x$ which is stabilized by $GL_n(\mathscr{O}_x)$.

On an unramified place we can define a *spherical Hecke algebra* $\mathscr{H}_x$. The is the space of compactly supported functions on the double-quotient $GL_n(\mathscr{O}_x)\backslash GL_n(F_x)/GL_n(\mathscr{O}_x)$. They have a multiplication give by convolution

$$f_1 \star f_2(g) = \int_{GL_n(F_x)} f_1(gh^{-1}) f_2(h) dh,$$

where dh is a Haar measure on $GL_n(F_x)$, normalized such that $GL_n(\mathscr{O}_x)$ has volume 1.

We can give a more explictly description of $\mathscr{H}_x$. Let $H_{i,x}$ denote the characteristic function of the double coset $GL_n(\mathscr{O}_x)\text{diag}(t_x, \dots, t_x, 1, \dots, 1)GL_n(\mathscr{O}_x)$. Where we have the local parameter t_x on the diagonal up to the i-th spot. The fact here is that $\mathscr{H}_x$ is the free $\mathbb{C}-$algebra on these elements, i.e.

$$\mathscr{H}_x \cong \mathbb{C}[H_{1,x}, \dots, H_{n-1,x}, H^{\pm}_{n,x}.$$

These $H_{i,x}$ are the so-called Hecke operators.

We also have an action of $\mathscr{H}_x$ on π_x, for π_x a representation of $GL_n(F_x)$. Let $v \in \pi_x$, an element $f_x \in \mathscr{H}_x$ acts by

$$f_x \star v = \int_{GL_n(F_x)} f_x(g) g \cdot v dg.$$

[6] We actually want to restrict to automorphic representations of GL_n which are not obtained by induction from representation of GL_n for a smaller n. This condition is expressed as the vanishing of the integral of f over any coset relative to a parabolic subgroup.

If we have an element $h \in GL_n(\mathscr{O}_x)$ and v is $GL_n(\mathscr{O}_x)$-invariant, then $f_x \star v$ is still $GL_n(\mathscr{O})$-invariant. The important fact here is that for an irreducible and unramified π_x, $\dim \pi_x^{GL_n(\mathscr{O}_x)} = 1$. So if we choose a certain nonzero element $v_x \in \pi_x$, invariant under $GL_n(\mathscr{O}_x)$ we get that all the operators $H_{i,x}$ act by multiplication by a complex number

$$H_{i,x} \star v_x = \phi(H_{i,x}) v_x.$$

And these numbers $\phi(H_{i,x})$ are called the Hecke eigenvalues of the the irreducible representation π_x.

Any homomorphism $\mathscr{H}_x \to \mathbb{C}$ can be recovered form the collection of $\{\phi(H_{1,x}), \ldots, \phi(H_{n,x})\}$. One can combine them in a symmetric manner so as to have a direct relation with certain eigenvalues on the Galois side of the Langlands correpondence, i.e. the Frobenius eigenvalues of a Galois representation. And an important result is that the Hecke eigenvalues at all unramified places allow one to reconstruct the whole autmorphic representation π up to isomorphism.