# Math 131-H – Homework 1 Solutions

1. (a) We know that $\log(x) \leq x - 1$ for all $x > 0$. In particular, we can replace $x$ by $1/x$, because when $x$ is greater than 0, $1/x$ is also greater than 0. Therefore

$$\begin{aligned} \log(1/x) &\leq 1/x - 1 \\ \implies -\log(x) &\leq 1/x - 1 \\ \implies \log(x) &\geq 1 - 1/x. \end{aligned}$$

(b) From part (a) we know that $1 - 1/x \leq \log(x) \leq x - 1$ for all $x > 0$. We'll use this to find upper and lower bounds for the function $\log(1 + x)/x$. First

$$\begin{aligned} \log(x) &\leq x - 1 \\ \implies \log(1 + x) &\leq (1 + x) - 1 = x \\ \implies \frac{\log(1 + x)}{x} &\leq x/x = 1. \end{aligned}$$

That's an upper bound. Using the other inequality:

$$\begin{aligned} \log(x) &\geq 1 - 1/x \\ \implies \log(1 + x) &\geq 1 - \frac{1}{1 + x} \\ &= \frac{1 + x}{1 + x} - \frac{1}{1 + x} \\ &= \frac{x}{1 + x} \\ \implies \frac{\log(1 + x)}{x} &\geq \frac{1}{1 + x}. \end{aligned}$$

So we've shown that

$$\frac{1}{1 + x} \leq \frac{\log(1 + x)}{x} \leq 1$$

for all $x > 0$. To apply the squeeze theorem, we only need to observe that the two limits $\lim_{x \to 0} \frac{1}{1+x}$ and $\lim_{x \to 0} 1$ are both equal to 1 – the squeeze theorem then tells us that $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$ also.

2. (a) We will use the fact that we were given: that there exists a positive integer $n$ such that $b - a > 1/n$. That means that $nb - na > 1$, or to say it another way, the interval $(na, nb)$ has length greater than 1. Because the distance between any two consecutive integers on the number line is 1, and the interval $(na, nb)$ has length greater than 1, that means there is some integer $m$ in the interval $(na, nb)$.

Finally, because $m$ is in $(na, nb)$, that means $m/n$ is in $(a, b)$. The number $m/n$ is rational, so we are done.

(b) We just need to notice that a number $y$ is in the interval $(a, b)$ if and only if the number $y - \sqrt{2}$ is in the interval $(a - \sqrt{2}, b - \sqrt{2})$. To put it another way, a number $x + \sqrt{2}$ is in the interval $(a, b)$ if and only if the number $x$ is in the interval $(a - \sqrt{2}, b - \sqrt{2})$.

By part (a), there is a rational number $x$ in any interval. In particular there is a rational number $x$ in the interval $(a - \sqrt{2}, b - \sqrt{2})$. Therefore the number $x + \sqrt{2}$ is in the interval $(a, b)$ for any $a < b$. We're given that $x + \sqrt{2}$ is irrational, so we are done.

(c) Let $x_0$ be any real number: we will show that the limit of $i(x)$ does not exist at $x = x_0$. We let $\varepsilon = 1/2$, and we'll show that $f(x)$ cannot be made to stay within $\varepsilon$ of any number: no matter how small an interval around $x_0$ we look at.

For any positive number $\delta$, the interval $(x_0, x_0 + \delta)$ contains a rational number $r$ by part (a), and it contains an irrational number $s$ by part (b). That means that in the interval $(x_0, x_0 + \delta)$ there is a value $r$ where $i(r) = 1$, and a value $s$ where $i(s) = 0$. So the function $i$ does not remain within $\varepsilon = 1/2$ of any value, and therefore the limit does not exist.