# Math 131-H – Homework 2 Solutions

1. (a) The derivative of the function $f(x) = x^2/4a$ is $f'(x) = x/2a$, so the tangent line at the point $(x_0, f(x_0))$ has equation

$$y - \frac{x_0^2}{4a} = \frac{x_0}{2a}(x - x_0)$$
$$y = \frac{x_0}{2a}a - \frac{x_0^2}{4a},$$

with $y$-intercept $-\frac{x_0^2}{4a}$.

(b) First we'll compute the square of the distance from $(x_0, f(x_0))$ to the focus $(0, a)$:

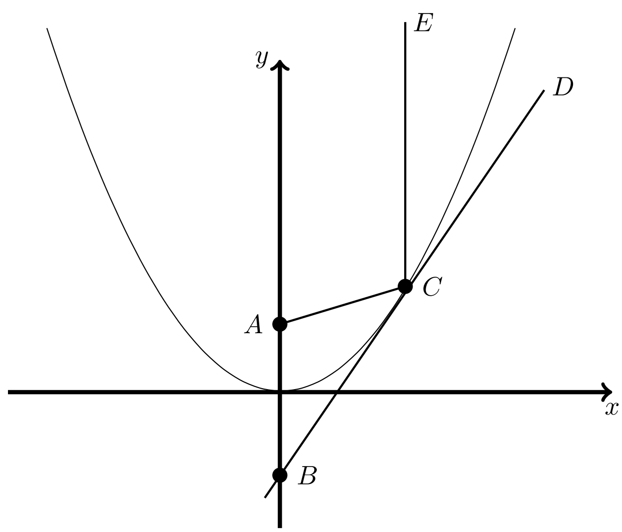
$$\begin{aligned} x_0^2 + (f(x_0) - a)^2 &= x_0^2 + \left(\frac{x_0^2}{4a} - a\right)^2 \\ &= x_0^2 + \frac{x_0^4}{16a^2} - \frac{x_0^2}{2} + a^2 \\ &= \frac{x_0^4}{16a^2} + \frac{x_0^2}{2} + a^2. \end{aligned}$$

Now we'll compute the square of the distance from from the $y$-intercept $(0, -\frac{x_0^2}{4a})$ to the focus $(0, a)$. It is nothing but

$$\left(a + \frac{x_0^2}{4a}\right)^2 = \frac{x_0^4}{16a^2} + \frac{x_0^2}{2} + a^2,$$

so the two distances are equal, and the triangle is isosceles.

(c) Consider the following picture, where the line through $B$ and $D$ is the tangent line to the curve at $(x_0, f(x_0))$:



Because the triangle $ABC$ is isosceles, the angles $\mathrm{Angle}(ABC)$ and $\mathrm{Angle}(BCA)$ are equal. Because the lines $AB$ and $EC$ are parallel (they are both vertical lines), the angles $\mathrm{Angle}(ABC)$ and $\mathrm{Angle}(ECD)$ are also equal. Therefore the angles $\mathrm{Angle}(BCA)$ and $\mathrm{Angle}(ECD)$ are equal, which is the reflection property that we were looking for.

2. Let's start by finding the equation for a tangent line to the curve $y = 1/x$ at a point on the curve, say the point $(w, 1/w)$ for some $w$. The derivative of $1/x$ is $-1/x^2$, so the tangent line has equation

$$y - 1/w = -1/w^2(x - w)$$
$$\text{or } y = -x/w^2 + 2/w.$$

Now, take a fixed point $(x_0, y_0)$ somewhere in the plane. This point lies on the tangent line to the curve at $(w, 1/w)$ if

$$y_0 = -x_0/w^2 + 2/w$$
$$\text{or } y_0 w^2 = -x_0 + 2w$$
$$\text{or } y_0 w^2 - 2w + x_0 = 0.$$

We need to figure out how many tangent lines pass through the point $(x_0, y_0)$, which means, how many solutions does this equation have, when we solve for $w$? This is a quadratic equation in $w$, so it has solutions

$$w = \frac{1}{y_0}(1 \pm \sqrt{1 - x_0 y_0}).$$

(a) If $x_0 y_0 > 1$, then $1 - x_0 y_0 < 0$, so there are no real solutions to the equation for $w$, and no tangent lines through the point $(x_0, y_0)$.

(b) If $x_0 y_0 = 1$ then $1 - x_0 y_0 = 0$, so there is one solution for $w$, namely $w = 1/y_0$. So there is one tangent line through the point, with equation

$$y = -y_0^2 x + 2y_0$$
$$\text{or } y = -x/x_0^2 + 2/x_0.$$

(c) If $0 < x_0 y_0 < 1$ then $0 < 1 - x_0 y_0 < 1$, so there are two solutions for $w$. Therefore there are two tangent lines through the point, with equation

$$y = -x/w^2 + 2/w,$$

where $w = \frac{1}{y_0}(1 \pm \sqrt{1 - x_0 y_0})$.

(d) Here either $x_0$ or $y_0$ is equal to 0, which means instead of a quadratic equation for $w$, we have a linear equation for $w$, either $w = x_0/2$ or $w = y_0/2$. So there is one tangent line through the point $(x_0, y_0)$ with equation

$$y = -4x/x_0^2 + 1/x_0$$

or

$$y = -4x/y_0^2 + 1/y_0.$$

(e) If $x_0 = y_0 = 0$ then the equation becomes $w = 0$. Since the function $f(x)$ is not defined at $x = 0$, there is no tangent line through $(0, 0)$.

(f) Finally, if $x_0 y_0 < 0$ then $1 - x_0 y_0 > 1$, so there are two solutions for $w$, therefore there are two tangent lines through the point, with the same equation as in (c).