Math 131-H – Homework 3 Solutions

1. (a) Using the product rule, we find

$$\begin{aligned} H_1(x) &= -x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= -x^3 + 3x \\ H_4(x) &= x^4 - 6x^2 + 3. \end{aligned}$$

   (b) Use the product rule again. So

$$\begin{aligned} H_n(x)e^{-x^2/2} &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{-x^2/2}) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(H_{n-1}(x)e^{-x^2/2}) \\ &= (H'_{n-1}(x) - xH_{n-1}(x))e^{-x^2/2} \\ \text{so } H_n(x) &= H'_{n-1}(x) - xH_{n-1}(x). \end{aligned}$$

2. (a) Use the distance formula. So the Oval of Cassini is given by the equation

$$((x-1)^2 + y^2)^{1/2}((x+1)^2 + y^2)^{1/2} = b^2$$

   or, squaring both sides, by

$$((x-1)^2 + y^2)((x+1)^2 + y^2) = b^4.$$

   Expanding out the left-hand side we find

$$\begin{aligned} ((x-1)^2 + y^2)((x+1)^2 + y^2) &= (x^2 + y^2 - 2x + 1)(x^2 + y^2 + 2x + 1) \\ &= (x^2 + y^2)^2 - 2x(x^2 + y^2 + 2x + 1) + 2x(x^2 + y^2 + 1) \\ &\quad + (x^2 + y^2 - 2x + 1) + (x^2 + y^2 + 2x) \\ &= (x^2 + y^2)^2 - 4x^2 + 2x^2 + 2y^2 + 1) \\ &= (x^2 + y^2)^2 - 2(x^2 - y^2) + 1 \end{aligned}$$

   which is the required expression.

   (b) This is the same as we did in class. We find the slope of the tangent line using implicit differentiation. So

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}((x^2 + y^2)^2 - 2(x^2 - y^2) + 1) &= \frac{\mathrm{d}}{\mathrm{d}x}b^4 \\ \implies 2(2x + 2yy')(x^2 + y^2) - 4x + 4yy' &= 0 \\ \implies 4(1 + (x^2 + y^2))yy' &= 4(1 - (x^2 + y^2))x \\ \implies y' &= \frac{(1 - (x^2 + y^2))x}{(1 + (x^2 + y^2))y}. \end{aligned}$$

   The tangent line to the curve at a point (x_0, y_0) therefore has equation

$$(y - y_0) = \frac{(1 - (x^2 + y^2))x}{(1 + (x^2 + y^2))y}(x - x_0).$$

(c) The tangent line is horizontal when the slope is zero, i.e. when $(1-(x^2+y^2))x=0$, so either $x^2+y^2=1$ or $x=0$. Let's suppose $x=0$ first. If a point on the curve has $x=0$, plugging this into the equation for the curve we get

$$\begin{aligned} y^4+2y^2+1 &= b^4 \\ \text{or } u^2+2u+(1-b^4) &= 0 \text{ for } u=x^2 \\ \implies u &= -1\pm\sqrt{b^4} \\ &= -1\pm b^2 \\ \text{so } y = \pm\sqrt{-1\pm b^2}. \end{aligned}$$

So the tangent line is horizontal at $(x,y)=(0,\pm\sqrt{-1\pm b^2})$.

Now, take the other possibility. If a point on the curve has $(1-(x^2+y^2))x=0$, plugging this into the equation for the curve we get

$$\begin{aligned} 1-2(1-2y^2)+1 &= b^4 \\ 4y^2 &= b^4 \\ \implies y &= \pm\frac{b^2}{2} \\ \text{and } x &= \pm\sqrt{1-y^2} \\ &= \pm\sqrt{1-\frac{b^4}{4}}. \end{aligned}$$

So the tangent line is also horizontal at $(x,y)=\left(\pm\sqrt{1-\frac{b^4}{4}},\pm\frac{b^2}{2}\right)$.

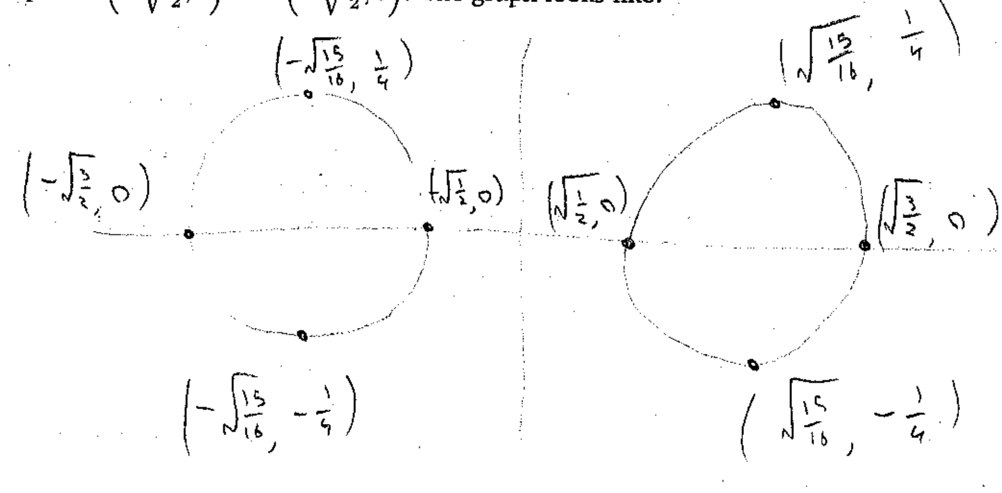
The tangent line is vertical when the slope goes to infinity, i.e. when $(1+(x^2+y^2))y=0$, so wither $x^2+y^2=-1$ or $y=0$. The former never happens, so we only need to consider the latter. If a point on the curve has $y=0$, plugging this into the equation for the curve we get

$$\begin{aligned} x^4-2x^2+1 &= b^4 \\ \text{or } u^2-2u+(1-b^4) &= 0 \text{ for } u=x^2 \\ \implies u &= 1\pm\sqrt{b^4} \\ &= 1\pm b^2 \\ \text{so } x = \pm\sqrt{1\pm b^2}. \end{aligned}$$

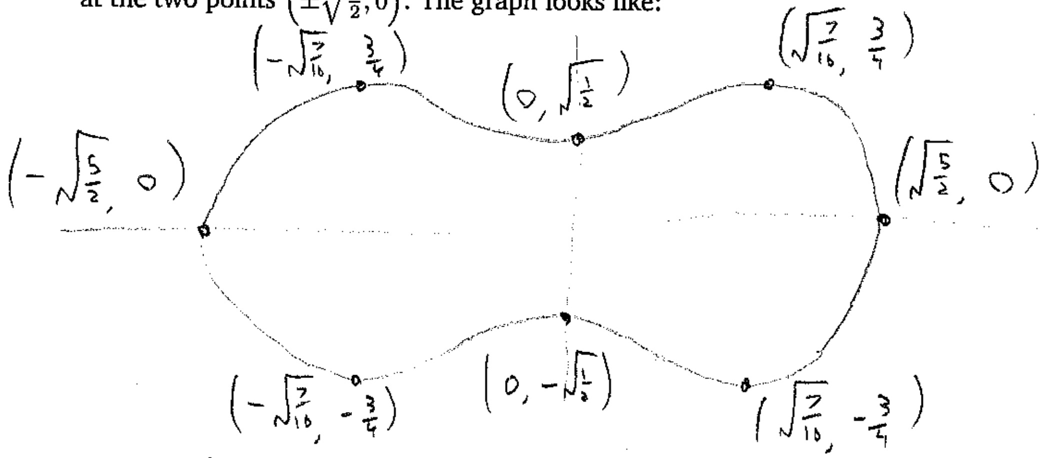
So the tangent line is vertical at $(x,y)=(\pm\sqrt{1\pm b^2},0)$.

(d) We can sketch the graphs in each case by noting the points where the tangent line is horizontal and vertical, and connecting these points together.

- If $b^2=0.5$, the tangent line is horizontal at the four points $\left(\pm\sqrt{\frac{15}{16}},\pm\frac{1}{4}\right)$. It is vertical at the four points $\left(\pm\sqrt{\frac{1}{2}},0\right)$ and $\left(\pm\sqrt{\frac{3}{2}},0\right)$. The graph looks like:



- If $b^2 = 1.5$, the tangent line is horizontal at the six points $\left(0, \pm\sqrt{\frac{1}{2}}\right)$ and $\left(\pm\sqrt{\frac{7}{16}}, \pm\frac{3}{4}\right)$. It is vertical at the two points $\left(\pm\sqrt{\frac{5}{2}}, 0\right)$. The graph looks like:



- Finally, if $b^2 = 3$, the tangent line is horizontal at the two points $(0, \pm\sqrt{2})$ and vertical at the two points $(\pm 2, 0)$. The graph looks like: