Math 131-H – Homework 4 Solutions

1. (a) Using the quotient rule,

$$\tanh(x)' = (\cosh(x)^2 - \sinh(x)^2)/\cosh^2(x) = \text{sech}^2(x).$$

Now, by implicit differentiation

$$\begin{aligned} y &= \text{arctanh}(x) \\ x &= \tanh(y) \\ \frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(\tanh(y)) \\ 1 &= y'\text{sech}^2(y) \\ \text{so } y' &= 1/\text{sech}^2(y) \\ &= 1/\text{sech}^2(\text{arctanh}(x)) \\ &= \frac{1}{1-\tanh(\text{arctanh}(x))^2} \\ &= \frac{1}{1-x^2}. \end{aligned}$$

(b) Similarly, using the quotient rule

$$\coth(x)' = (\sinh(x)^2 - \cosh(x)^2)/\sinh^2(x) = -\text{csch}^2(x).$$

Now, by implicit differentiation

$$\begin{aligned} y &= \text{arccoth}(x) \\ x &= \coth(y) \\ \frac{\mathrm{d}x}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}(\coth(y)) \\ 1 &= -y'\text{csch}^2(y) \\ \text{so } y' &= -1/\text{csch}^2(y) \\ &= -1/\text{csch}^2(\text{arccoth}(x)) \\ &= -\frac{1}{\coth(\text{arccoth}(x))^2-1} \\ &= \frac{1}{1-x^2}. \end{aligned}$$

(c) Notice that these two functions are equal, yet $\text{arctanh}(x)$ and $\text{arccoth}(x)$ don't differ by a constant. The reason is that $\text{arctanh}(x)$ is only defined for $-1 < x < 1$ and $\text{arccoth}(x)$ is only defined for $x > 1$ and $x < -1$, so the domains of definition of the two functions don't overlap.

2. (a) If $\zeta = \text{arctanh}(v/c)$ then $v = c\tanh(\zeta)$. Plug this into the equations for x_{new} and t_{new}. First, we calculate that

$$\gamma(v) = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\tanh^2(\zeta)}} = \frac{1}{\text{sech}(\zeta)} = \cosh(\zeta).$$

Now, we get

$$\begin{aligned} x_{\text{new}} &= \cosh(\zeta)(x + c\tanh(\zeta)t) \\ &= \cosh(\zeta)x + c\sinh(\zeta)t, \\ t_{\text{new}} &= \cosh(\zeta)(t + \tanh(\zeta)x/c) \\ &= \cosh(\zeta)t + \sinh(\zeta)x/c. \end{aligned}$$

(b) Using the chain rule, we calculate

$$\begin{aligned} u_{\text{new}} &= \frac{\mathrm{d}x_{\text{new}}}{\mathrm{d}t_{\text{new}}} \\ &= \frac{\mathrm{d}x_{\text{new}}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t_{\text{new}}} \\ &= \frac{\mathrm{d}x_{\text{new}}}{\mathrm{d}t}\left(\frac{\mathrm{d}t_{\text{new}}}{\mathrm{d}t}\right)^{-1}. \end{aligned}$$

So, taking those terms individually, and using that v and c are constant,

$$\begin{aligned} \frac{\mathrm{d}x_{\text{new}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(v)(x + vt)) \\ &= \gamma(v)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + v\frac{\mathrm{d}t}{\mathrm{d}t}\right) \\ &= \gamma(v)(u + v), \\ \text{and } \frac{\mathrm{d}t_{\text{new}}}{\mathrm{d}t} &= \gamma(v)\left(\frac{\mathrm{d}t}{\mathrm{d}t} + v/c^2\frac{\mathrm{d}x}{\mathrm{d}t}\right) \\ &= \gamma(v)(1 + uv/c^2). \end{aligned}$$

So altogether,

$$u_{\text{new}} = \frac{u + v}{1 + uv/c^2}.$$

(c) If we differentiate u_{new} with respect to u we get

$$\frac{\mathrm{d}u_{\text{new}}}{\mathrm{d}u} = \frac{(1 + uv/c^2) - v/c^2(u + v)}{(1 + uv/c^2)^2},$$

so if we plug in $v = c$ we're left with $\frac{1+u/c-u/c-1}{(1+u/c)^2} = 0$. Alternatively, and more straightforwardly, we can plug in $v = c$ initially, to get

$$u_{\text{new}} = \frac{u + c}{1 + u/c} = \frac{c(u + c)}{c + u} = c,$$

which is constant, so its derivative is automatically zero.