Math 131-H – Homework 5 Solutions

1. (a) To find the critical points of $f(x)$, we differentiate to find

$$f'(x) = 6x^2 - 2bx + c.$$

This quadratic function has roots at the points where $x = \frac{b\pm\sqrt{b^2-6c}}{6}$. For there to be two roots, we need the expression inside the square root to be positive, so $b^2 > 6c$.

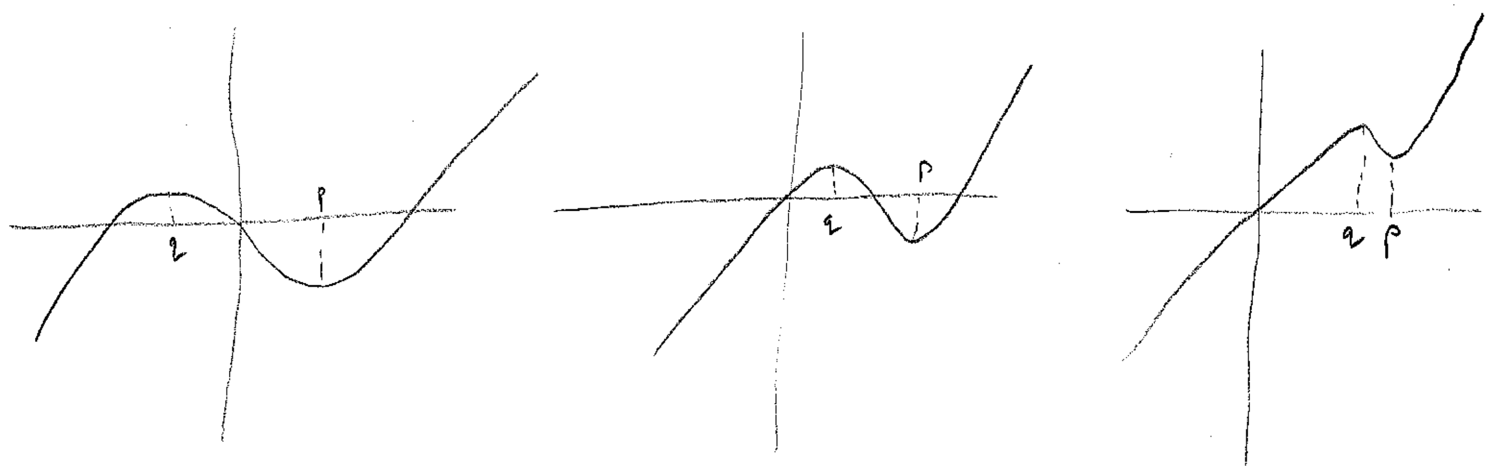
We need to check that one point is a local minimum and one is a local maximum (and see which is which!) We can do that using the second derivative:

$$f''(x) = 12x - 2b.$$

Plugging in our two roots, we find that $f''(\frac{b\pm\sqrt{b^2-6c}}{6} = \pm 2\sqrt{b^2-6c}$. So the point $q = \frac{b-\sqrt{b^2-6c}}{6}$ is a local maximum, and the point $p = \frac{b+\sqrt{b^2-6c}}{6}$ is a local minimum.

(b) Using our expression for the second derivative above, $f''(x) = 12x - 2b$, we observe that $f''(x)$ is positive when $x > b/6$, and so the function is concave upwards on that region, and $f''(x)$ is negative when $x < b/6$, so the function is concave downwards on that region. The point $x = b/6$ is an inflection point, where the function changes concavity.

(c) The function $f(x)$ is defined everywhere, and $f(x) \to \pm\infty$ as $x \to \pm\infty$. The last thing we need to before graphing is to find the roots. We can factorize $f(x) = x(2x^2 - bx + c)$, and so the roots occur at $x = 0$ and $x = \frac{b\pm\sqrt{b^2-8c}}{4}$. Any of the following graphs are acceptable.



2. (a) For the function to be defined, we need the denominator to be non-zero, and we need the expression inside the square-root to be non-negative. So overall, the function is defined for those x where $x^2 - 2x + a > 0$. If you look at the graph of a quadratic function, you can see that its y-value is less than zero in between the two roots. So we find the roots using the quadratic formula. They occur when $x = 1 - \sqrt{1-a}$ and $x = 1 + \sqrt{1-a}$. If $a < 1$ then the quadratic has no roots, so it is positive everywhere, and $f(x)$ is defined everywhere. If, however, $a \geq 1$, then the function $f(x)$ is only defined when $x < 1 - \sqrt{1-a}$ and when $x > 1 + \sqrt{1-a}$.

(b) We compute the derivative of $f(x)$ using the quotient rule:

$$f'(x) = \frac{(x^2-2x+a)^{1/2} - \frac{1}{2}x(2x-2)(x^2-2x+a)^{-1/2}}{x^2-2x+a}$$
$$= \frac{(x^2-2x+a) - \frac{1}{2}x(2x-2)}{(x^2-2x+a)^{3/2}}$$
$$= \frac{a-x}{(x^2-2x+a)^{3/2}}.$$

So $f'(x) = 0$ only when $x = a$. This point is in the region where $f(x)$ is defined as long as $a < 1$.

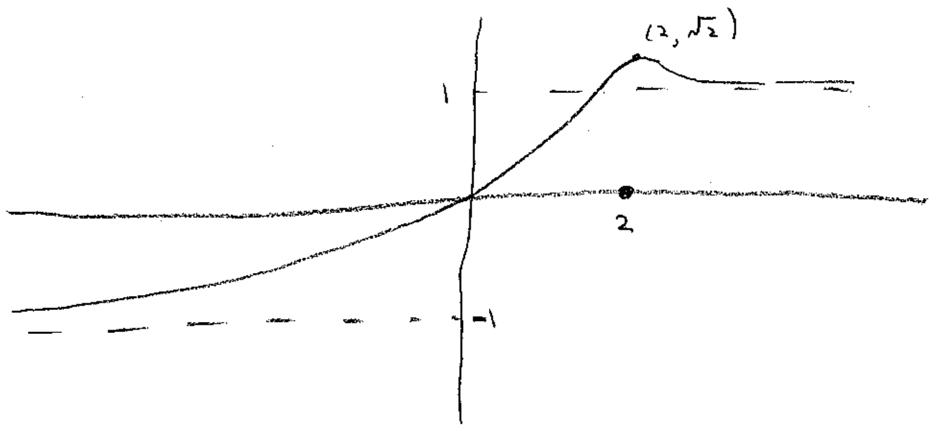
(c) We'll use the limit laws. If $x > 0$ then $x = \sqrt{x^2}$, so we have:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-2x+a}}$$
$$= \lim_{x\to\infty} \frac{x/x}{(\sqrt{x^2-2x+a})/x} = \lim_{x\to\infty} \frac{1}{(\sqrt{x^2-2x+a})/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-2x^{-1}+ax^{-2}}} = 1.$$

On the other hand, if $x < 0$ then $x = -\sqrt{x^2}$, so the calculation changes in the following way:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-2x+a}}$$
$$= \lim_{x\to-\infty} \frac{x/x}{(\sqrt{x^2-2x+a})/x} = \lim_{x\to-\infty} \frac{1}{(\sqrt{x^2-2x+a})/(-\sqrt{x^2})} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1-2x^{-1}+ax^{-2}}} = -1.$$

(d) As well as what we've already observed, we note that $f(0) = 0$ is the only root.



(e) Again, $f(0) = 0$ is the only root. Note here that $f(x) = \frac{x}{|x-1|}$, which has a vertical asymptote at $x = 1$, and $\lim_{x\to 1} f(x) = \infty$.