Math 131-H – Homework 6 Solutions

1. For this problem, we need to think of the integral as a function of c, so define

$$G(c) = \int_a^b (f(x) - c)^2 \mathrm{d}x.$$

We need to find the global minimum of the function $G(c)$. So let's find the critical points. So, compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}c} G(c) &= \frac{\mathrm{d}}{\mathrm{d}c} \int_a^b (f(x)^2 - 2cf(x) + c^2 \mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}c} \left(\int_a^b f(x)^2 \mathrm{d}x \right) - 2\frac{\mathrm{d}}{\mathrm{d}c} \left(c \int_a^b f(x) \mathrm{d}x \right) + \frac{\mathrm{d}}{\mathrm{d}c} \left(c^2 \int_a^b \mathrm{d}x \right) \\
&= 0 - 2 \int_a^b f(x) \mathrm{d}x + 2c \int_a^b \mathrm{d}x \\
&= -2 \int_a^b f(x) \mathrm{d}x + 2c(b - a).
\end{aligned}$$

So there is a single critical point at $c = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x$. If we differentiate again, we find $\mathrm{d}^2 \mathrm{d}c^2 G(c) = 2(b - a)$, which is positive, so this critical point is indeed a minimum.

2. (a) Choose some x between 0 and 1. The mean value theorem says that there is some number c between 0 and x, so that

$$e^c = \frac{e^x - 1}{x}.$$

Because c was between 0 and 1, that means

$$1 \leq \frac{e^x - 1}{x} \leq e < 3$$

and so

$$x + 1 \leq e^x < 3x + 1.$$

(b) By part (a), we know

$$\int_0^1 (x + 1) \mathrm{d}x \leq \int_0^1 e^x \mathrm{d}x \leq \int_0^1 (3x + 1) \mathrm{d}x.$$

The integrals on the outside are easy to evaluate by thinking about the area as the combination of a rectangle and a triangle, so we get that

$$3/2 \leq \int_0^1 e^x \mathrm{d}x \leq 5/2,$$

which implies that

$$1 < \int_0^1 e^x \mathrm{d}x < 3.$$

Now, that's great for the lower bound, but we need something better to get the upper bound. Because $f(x) = e^x$ is concave upwards, on the interval $[0, 1]$ the graph of e^x lies below the straight line from $(0, 1)$ to $(1, e)$. The area under this straight line is $1 + \frac{e-1}{2}$, so

$$\int_0^1 e^x \mathrm{d}x \leq 1 + \frac{e - 1}{2} < 2.$$

3. (a) We use the hint. First, note that $\sum_{k=0}^{n-1}(k+1)^3 - k^3 = n^3$ – all other terms cancel. So

$$\begin{aligned} n^3 &= \sum_{k=0}^{n-1}(k+1)^3 - k^3 \\ &= \sum_{k=0}^{n-1} 3k^2 + 3k + 1 \\ &= 3\sum_{k=0}^{n-1} k^2 + 3\sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 \\ &= 3\sum_{k=0}^{n-1} k^2 + \frac{3n(n-1)}{2} + n \\ \text{so } \sum_{k=0}^{n-1} k^2 &= \frac{n^3}{3} - \frac{n(n-1)}{2} - \frac{n}{3} \\ &= \frac{n(n-1)(2n-1)}{6}. \end{aligned}$$

(b) Again, we use the hint.

$$\begin{aligned} n^4 &= \sum_{k=0}^{n-1}(k+1)^4 - k^4 \\ &= \sum_{k=0}^{n-1} 4k^3 + 6k^2 + 4k + 1 \\ &= 4\sum_{k=0}^{n-1} k^3 + 6\sum_{k=0}^{n-1} k^2 + 4\sum_{k=0}^{n-1} k + \sum_{k=0}^{n-1} 1 \\ &= 4\sum_{k=0}^{n-1} k^3 + n(n-1)(2n-1) + n(n-1) + n \\ \text{so } \sum_{k=0}^{n-1} k^3 &= \frac{n^4 - n(n-1)(2n-1) - n(n-1) - n}{4} \\ &= \frac{n^2(n-1)^2}{4}. \end{aligned}$$

(c) Using the definition of the integral as a limit of (left) Riemann sums, we get that

$$\begin{aligned} \int_0^2 f(x)\mathrm{d}x &= \lim_{n\to\infty} \sum_{k=0}^{n-1} f(\frac{2k}{n})\frac{2}{n} \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{16k^3}{n^4} + \frac{8k^2}{n^3} \\ &= \lim_{n\to\infty} \frac{4n^2(n-1)^2}{n^4} + \frac{4n(n-1)(2n-1)}{3n^3} \\ &= 4 + 8/3 = 20/3. \end{aligned}$$