Line and Surface Operators in Gauge Theories

Chris Elliott

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1 Wilson and 't Hooft Operators

1.1 Wilson Operators as Holonomy

Consider Yang-Mills theory for a compact connected Lie group G on a Riemannian manifold X (for now we can get away with assuming very little about X. In particular we needn't assume that X is 4-dimensional). Fields in such a theory are connections A on principal G-bundles $P \to X$. A natural thing we might try to observe in the theory is the *holonomy* of A around curves $\gamma \subseteq X$. Let's recall what this means.

So let $\gamma \subseteq X$ be an oriented closed curve, and choose a base-point $x \in \gamma$. A connection A defines parallel transport in the bundle P; in particular, parallel transport around γ defines an automorphism $\alpha_{A,\gamma}$ of the fibre P_x at x, hence an element $\mathrm{Hol}_\gamma(A) = \alpha_{A,\gamma}(e)$ of the group G: the *holonomy* of the connection around the curve γ. This element is independent of the choice of base-point x up to conjugation, and reversing the orientation on γ sends $\alpha_{A,\gamma}(e)$ to its inverse.

Now, we'd like to produce a *number* from the data we chose. To do this, choose a finite-dimensional real or complex representation ρ of G. We define the *Wilson operator* around γ to be the observable

$$W_{\gamma,\rho} \colon A \mapsto \mathrm{Tr}(\rho(\mathrm{Hol}_\gamma(A))).$$

Note that this is a gauge invariant observable: performing a gauge transformation conjugates $\mathrm{Hol}_\gamma(A)$ by an element of the group, which doesn't affect the observable by cyclic invariance of the trace.

1.2 Exponential description

We can produce a more algebraic construction of this Wilson operator, which we can manipulate more easily. This will be easiest to understand first in the abelian case, so let's temporarily fix $G = U(1)$. We may as well take ρ to be 1-dimensional, so let ρ be the representation $z \mapsto z^n$ for $n \in \mathbb{Z}$. We'll argue that in this case

$$\mathrm{Hol}_\gamma(A) = e^{i \oint_\gamma A}$$

and so

$$W_{\gamma,n}(A) = e^{in \oint_\gamma A}.$$

Let's think about what this means. The gauge field A is not actually a 1-form, but upon choosing a principal $U(1)$-bundle the connections on that bundle become a *torsor* for $\Omega^1(X)$. So we choose a base connection A_P on each $U(1)$-bundle. We may as well choose these base connections by choosing a flat connection on the trivial bundle, and translating this trivialisation to the other bundles canonically by adding an appropriate harmonic form. The result will be independent of exactly which flat connection we choose. Given this, we can define

$$W_{\gamma,n}(A) = e^{in \oint_\gamma (A - A_P)}$$

for a completely well-defined expression.

Remark 1.1. If the curve γ is a *boundary*, i.e. if the class $[\gamma] = 0$ in the first homology group of X, then we have a nice description of the Wilson operator by Stokes' theorem. If γ bounds a disc D then we can write the Wilson operator as

$$W_{\gamma,n}(A) = e^{in\int_D F_A}.$$

Since F_A is a genuine 2-form, this expression is already well-defined. We also observe that it is independent of the choice of the bounding disc D. In this situation we still have a well-defined operator even if n is not an integer.

I should at least remark on why this definition agrees with the previous one, i.e. why this exponential computes the holonomy. Consider the restriction of a circle bundle P to γ. This restriction is trivial, so choose an explicit trivialisation $P|_\gamma \cong \gamma \times U(1)$. The metric on X gives us a special vector field on γ, namely $\dot{\gamma}$, and the parallel transport equation is the requirement that a section σ satisfies

$$\nabla_{\dot{\gamma}(\theta)}\sigma = 0,$$

for θ a co-ordinate on $U(1)$. That is,

$$\iota_{\dot{\gamma}(\theta)} d\sigma = i\iota_{\dot{\gamma}(\theta)} A\sigma.$$

So plug in $\sigma(\theta) = e^{i\int_0^\theta A}$, and check that it satisfies the equation.

1.3 Path Ordering

Ok, what about *non-abelian* groups? There's still an exponential description in general, but it requires a little more care to define. Specifically, since we're dealing with the exponential of a matrix-valued function, we have to introduce a gadget called the *path-ordered exponential* to make sense of it. We can make a definition in local co-ordinates.

Definition 1.2. Given a matrix-valued function M on $\mathbb{R}$, we define the *path-ordered exponential* of M by a power series

$$\mathcal{P}e^{\int_0^t M(t_1)dt_1} = 1 + \int_0^t M(t_1)dt_1 + \int_0^t \int_0^{t_1} M(t_1)M(t_2)dt_2dt_1 + \int_0^t \int_0^{t_1} \int_0^{t_2} M(t_1)M(t_2)M(t_3)dt_3dt_2dt_1 + \cdots .$$

We can use this local definition to define the path-ordered exponential

$$\mathrm{Hol}_\gamma(A) = \mathcal{P}e^{i\oint_\gamma A}$$

and hence the Wilson operator $W_{\gamma,\rho}(A)$ (where we must trivialise the $\Omega^1(X)$-torsors on each bundle to make this well-defined, as before). With enough care, one can check that this expression does indeed define the holonomy of the connection around the curve γ.

1.4 't Hooft Operators

Now, recall that in the case where the curve γ bounds a disc D, we have a natural description of the Wilson operators that only involves the *curvature* of the fields. From the point of view of quantum physics, this is actually the most important case, for the following reason.

Proposition 1.3. If γ represents a non-trivial class in $H_1(X)$ then the expectation value $\langle W_{\gamma,\rho}\rangle$ is zero.

Proof sketch. Let me explain heuristically why this is true in the example where $G = U(1)$. We'll use gauge invariance of the expectation value. For any closed 1-form α we can perform an upper-triangular change of variables on the fields $A \mapsto A + \alpha$ without changing the expectation value. So in the path integral

$$\frac{1}{Z}\int DAe^{in\oint_\gamma A}e^{iS(A)} = \frac{1}{Z}\int DAe^{in\left(\oint_\gamma A + \oint_\gamma \alpha\right)}e^{iS(A)}$$

since the Yang-Mills action is preserved by addition of a closed form. Therefore $e^{in\oint_\gamma \alpha} = 1$, so $\oint_\gamma \alpha = 0$ (if it's non-zero we can multiply by a constant to ensure it's not a multiple of π). However, $\oint_\gamma \alpha$ is precisely the pairing between $[\gamma] \in H_1(X)$ and $[\alpha] \in H^1(X)$. This pairing is non-degenerate, so for the result to be zero for all $[\alpha]$, we require $[\gamma] = 0$. □

Let's assume now that $[\gamma] = 0$, so γ bounds a disc D. Furthermore, let's specify that $\dim X = 4$. In this case, there's another natural operator we might define that's supported on a neighbourhood of γ. First let's discuss the case $G = U(1)$.

Definition 1.4. The abelian *'t Hooft operator* $T_{\gamma,b}(A)$ is the observable

$$T_{\gamma,b} \colon A \mapsto e^{ib\int_D *F_A}.$$

Notice that this no longer can be expressed as the exponential integral of a 1-form: the integrand is no longer close to exact, but rather the sum of an integral harmonic form and a *coexact* form.

Let's discuss a more traditional description of the 't Hooft operators, again restricting ourselves to the abelian case for simplicity (though it's quite possible to define 't Hooft operators for non-abelian gauge groups, as I'll discuss later). One can describe 't Hooft operators in terms of what happens when they are inserted into a path integral. Specifically they have the effect of imposing a *singularity condition* on the fields around γ. This is quite easy to derive, by algebraic manipulation.

So consider a path integral involving an 't Hooft operator, and complete the square as follows:

$$\begin{aligned}\frac{1}{Z}\int DA\mathcal{O}(A)e^{ib\int_D *F_A}e^{i\int F_A\wedge *F_A} &= \frac{1}{Z}\int DA\mathcal{O}(A)e^{ib\int_D *F_A}e^{i\int(b\delta_D+F_A)\wedge *F_A}\\ &= \frac{1}{Z}e^{i\frac{b^2}{4}\int \delta_D\wedge *\delta_D}\int D\widetilde{A}\mathcal{O}(A)e^{i\int F_{\widetilde{A}}\wedge *F_{\widetilde{A}}}\end{aligned}$$

where δ_D is the 2-current Poincaré dual to $[D] \in C_2(X)$. We performed the change of variables $F_A \mapsto F_{\widetilde{A}} = F_A - \frac{b}{2}\delta_D$. Or – on the level of gauge fields – $A \mapsto \widetilde{A} = A - \frac{b}{2}d^{-1}\delta_D$ (we'll make precise what this means shortly). How should we interpret these new variables? We should think of $\widetilde{A}$ as a connection with a *simple pole* around the curve A. Indeed, there are ways steps to see this, by calculation in co-ordinates, and by studying the behaviour of the curvature after change of variables.

We can analyse the curvature by integrating it over various 2-cycles in X. If we take a small 2-sphere embedded in X, representing zero in $H_2(X)$, then we look at the integral $\int_{S^2} F_{\widetilde{A}} = -\frac{b}{2}\int_{S^2}\delta_D$, which computes the linking number of γ and the 2-sphere. So generally this is 0, unless the 2-sphere is contained in a fibre of the normal bundle to γ in X, in which case we get $-\frac{b}{2}$ (the residue of the pole).

Now, let's look at $\widetilde{A}$ in coordinates. Work locally near a point in γ, i.e. consider the manifold $\mathbb{R}\times\mathbb{R}^3$, with γ the curve $\mathbb{R}\times\{0\}$. One can define a 2-form on $\mathbb{R}\times(\mathbb{R}^3\setminus\{0\})$ by

$$F = -\frac{b}{2} *_3 d\left(\frac{1}{\|\boldsymbol{x}\|}\right)$$

where $\boldsymbol{x}$ are coordinates on $\mathbb{R}^3$. The singular 2-forms $F_{\widetilde{A}}$ above are precisely those that can locally be written as the sum of a smooth 2-form on $\mathbb{R}\times\mathbb{R}^3$ and a 2-form with simple singularity along γ of this form. To check this, we observe that a form is determined by its integral over closed 2-cycles, and indeed that all closed 2-cycles are either boundaries, or homologous to spheres in fibres of the normal bundle to γ. We can integrate the above form over such spheres, and check that it agrees with the calculation above.

This also tells us precisely what kind of mathematical objects $\widetilde{A}$ are: they are connections on principal $U(1)$ bundles on $X\setminus\gamma$ whose integrals over small 2-spheres normal to γ are $-\frac{b}{2}$.

Remark 1.5. We can define observables like Wilson and 't Hooft operators in many dimensions. Whenever one can identify the fields in a theory with differential forms (as we did here via the curvature map), one has analogous exponential integrals (possibly path-ordered) that one can perform, just by integrating p-forms over p-submanifolds with boundary. If the total dimension is $2p$, one can also integrate the Hodge star of fields over such p-cycles.

A nice example is the theory where fields are circle- (or torus-) valued functions on a 2-manifold. Such a map ϕ has a derivative $d\phi$ which is a 1-form, so can be integrated over intervals. If one takes the limit as an interval becomes very short, divided by the length of the interval, one recovers directional derivatives of the field ϕ.

Let's say something about how this extends to the general, non-abelian case. The utility of this description "inserting an 't Hooft operator corresponds to imposing a simple singularity in the fields along γ" is that it naturally extends to the non-abelian case. To do so, we must choose a cocharacter $\mu\colon U(1) \to G$. This defines a pullback map $\mu^*\colon \Omega^2(X;\mathfrak{g}) \to \Omega^2(X)$. We give a definition of what it means to *insert* a non-abelian 't Hooft operator into a path integral (so this is not fully precise).

Definition 1.6. An *'t Hooft operator* $T_{\gamma,\mu}$ for gauge group G and cocharacter μ is the observable that – when inserted into the path integral – modifies the fields to singular connections $\widetilde{A}$, such that $\mu^* F_{\widetilde{A}}$ is the curvature of a connection with simple singularity along γ as above.

One can check that this depends on μ only up to conjugacy. We can write non-Abelian 't Hooft operators in the form

$$T_{\gamma,\mu}\colon A \mapsto e^{i\int_D \mu^* * F_A}$$

analogous to the formula we gave for abelian 't Hooft operators above.

2 Surface Operators

What if, in the abelian gauge theory, instead of integrating F_A over discs, we built observables from the integral of F_A over *closed* 2-submanifolds S? Of course, this is only non-trivial if S is *not* homologous to zero. Similarly, for the abelian gauge theory on a 4-manifold we could integrate $*F_A$ over closed 2-submanifolds. These are usually referred to as *surface operators*. Let me say a little about the latter kind of operator (the so-called *disorder* operators).

So, in the abelian gauge theory on a 4-manifold, we consider the operator

$$\Sigma_{S,b}\colon A \mapsto e^{ib\int_S *F_A}$$

for $b \in \mathbb{R}$ and $S \subseteq X$ a closed surface. Insertion of such an operator into the path integral corresponds to performing a change of variables $F_A \mapsto F_{\widetilde{A}} = F_A - \frac{b}{2}\delta_S$, where much as before, δ_S is the 2-current Poincaré dual to the 2-cycle $[S]$. Now, we can compute the integral of such a current over 2-cycles: one finds

$$\int_T F_{\widetilde{A}} = \int_T F_A - \frac{b}{2}|S \cap T|,$$

where the intersection number is computed by perturbing S and T to be transverse, as usual.

Note: This section is not complete, but I'm hoping to add more details later.