Introduction to $N = 2$ Gauge Theory

Chris Elliott

January 13th, 2015

1 Introduction and Emphasis

The goal of this seminar is to understand the *Nekrasov partition function* introduced by Nekrasov in 2003 [Nek03]. This is a generating function for correlation functions in an $N = 2$ supersymmetric gauge theory, so my goal today is to explain what $N = 2$ gauge theory is and why we might find it interesting. We begin with the latter question, why is this interesting? There are two intertwined answers.

1. Firstly, $N = 2$ gauge theory has been a rich source of powerful *4-manifold invariants*, as well as an organising principle that has led to advances in actually computing these invariants in examples. The first invariants of this sort – Donaldson's polynomial invariants [Don83] – were originally inspired by ordinary $\mathrm{SU}(n)$ Yang-Mills theory. However, in a ground-breaking 1988 paper, Witten [Wit88] gave a construction of Donaldson invariants as expectation values of certain observables in a twist of $N = 2$ gauge theory (we'll talk more about this shortly). This opened the door to later work of Seiberg and Witten [SW94], who gave an alternative description of these invariants *also* coming from $N = 2$ gauge theory, using the fact that these observables in the twisted theory could be computed in an effective theory describing the far IR limit of the full gauge theory. The Seiberg-Witten invariants are more readibly computable than the Donaldson invariants, and the moduli space from which one computes the invariants has nicer properties (compactness, for instance).

2. Secondly, $N = 2$ gauge theory is a source of interesting *dualities* between quantum field theories, a particularly notable example of which is the *Alday-Gaiotto-Tachikawa (AGT) correspondence* [AGT10]. Associated to our 4d gauge theory there is a 2d conformal field theory, and to suitable observables in the 4d theory we can assign observables in the 2d theory whose expectation values agree. We can do this all continuously in certain parameters that exist on both sides. A comprehensive review of material around the AGT correspondence is [Rod13].

Before moving on to explaining what the $N = 2$ theory is, let me say approximately what the Nekrasov partition function is. The Nekrasov partition function $Z(a, \varepsilon_1, \varepsilon_2, q)$ is a stand-in for the partition function of the twisted $N = 2$ theory which one can define mathematically rigorously. The partition function of the twisted theory can be computed by a localization procedure as the volume of the moduli space of vacua, which in this case will be the moduli space of instantons. This space doesn't have a well-defined volume, but after a suitable compactification procedure one can define a volume for each connected component $\mathcal{M}_k$ depending on certain regularisation parameters $a, \varepsilon_1, \varepsilon_2$, and put these together into a generating function, which we might write as

$$Z(a, \varepsilon_1, \varepsilon_2, q) = \sum_{q \in \mathbb{Z}} q^k \mathrm{vol}(\mathcal{M}_k).$$

Introducing a setting in which these "volumes" make sense will be one of the main goals of the seminar.

2 The Supersymmetry Algebra

Let's begin by introducing the $N = 2$ supersymmetry *algebra*. In general, supersymmetric field theories are field theories with an action of a certain super Lie algebra. Any Euclidean field theory on $\mathbb{R}^4$ comes with a natural action of the Lie algebra of infinitesimal isometries of $\mathbb{R}^4$. Supersymmetry algebra actions extend this by asking for appropriate odd symmetries to act in addition to this. Historically, supersymmetric field theories were introduced because a classical no-go theorem of Coleman-Mandula demonstrated that quantum field theories on $\mathbb{R}^4$ could never admit the action of a larger Lie algebra extending this isometry algebra in a non-trivial way. However, if one is willing to pass to the larger category of *super* Lie algebras, more structured theories (i.e. theories with more symmetries) can be found.

Definition 2.1. The (Euclidean) *Poincaré algebra* $\mathcal{P}$ in dimension 4 is the semidirect product $\mathfrak{so}(4) \ltimes \mathbb{R}^4$ where $\mathfrak{so}(4)$ acts on $\mathbb{R}^4$ by the fundamental representation. The *complex Poincaré algebra* $\mathcal{P}_\mathbb{C}$ is its complexification $\mathfrak{so}(4;\mathbb{C}) \ltimes \mathbb{C}^4$.

Remark 2.2. Throughout this section we'll use the exceptional isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and its complexification, $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$.

Now, what do super Lie algebras extending the Poincaré algebra look like? For simplicity we'll focus on the complexified case. As a vector space, it'll take the form $\mathcal{P}_\mathbb{C} \oplus \Pi S$, where Π indicates that the vector space S is placed in odd degree. The bracket between even and odd elements makes S a representation of the algebra $\mathcal{P}_\mathbb{C}$, and the bracket between odd elements yields a symmetric pairing $S \otimes S \to \mathcal{P}_\mathbb{C}$. At this point we make certain restrictions for physical reasons: a theorem of Haag-Łopuszański-Sohnius [HŁS75] says that only certain fermionic extensions of $\mathcal{P}_\mathbb{C}$ can act globally on the state space of a quantum field theory. Thus we make the following definition.

Definition 2.3. A *super Poincaré algebra* is a super Lie algebra of the form $\mathcal{P}_\mathbb{C} \oplus \Pi S$ where S is a spinorial complex representation of $\mathfrak{so}(4;\mathbb{C})$, and where the bracket between two odd elements is given by a symmetric $\mathfrak{so}(4;\mathbb{C})$-equivariant pairing $\Gamma\colon S \oplus S \to \mathbb{C}^4$.

Up to isomorphism there are not very many possible choices for super Poincaré algebras. The only irreducible spinorial representations of $\mathfrak{so}(4;\mathbb{C})$ are the fundamental representations S_+ and S_- of the two $\mathfrak{sl}(2;\mathbb{C})$ summands so general spinorial representations are of the form $S_+^{N_+} \oplus S_-^{N_-}$. The only equivariant symmetric pairings (up to rescaling) are given by the natural isomorphism $\Gamma\colon S_- \otimes S_+ \to \mathbb{C}^4$, possibly applied separately to multiple copies of $S_+ \oplus S_-$. This in particular forces the restriction $N_+ = N_-$ onto the allowed representations. When people talk about $N = k$ supersymmetry, the N they are referring to is this value: the number of copies of the representation $S_+ \oplus S_-$ (the Dirac spinor representation).

Definition 2.4. The $N = 2$ *super Poincaré algebra* is the super Poincaré algebra associated to the spinorial representation $S_+ \otimes W \oplus S_- \otimes W^*$, where W is a two dimensional complex vector space, and where the pairing is given by the canonical pairing Γ combined with the evaluation pairing $W \otimes W^* \to \mathbb{C}$.

We often ask for theories not just to admit a super Poincaré action, but something slightly more.

Definition 2.5. The *R-symmetry algebra* of a super Poincaré algebra is the algebra of outer automorphisms which act trivially on the bosonic piece. Concretely, for the super Poincaré algebra associated to $S = S_+ \otimes W \oplus S_- \otimes W^*$, the R-symmetry algebra is $\mathfrak{gl}(W)$. The *Supersymmetry algebra* is the semidirect product algebra $\mathfrak{gl}(W) \ltimes (\mathcal{P}_\mathbb{C} \oplus \Pi S)$.

In the case of the $N = 2$ supersymmetry algebra, the R-symmetry algebra is just $\mathfrak{gl}(2;\mathbb{C})$.

2.1 Topological Supercharges

A little later we'll discuss twists of $N = 2$ supersymmetric gauge theories. While constructing these is a non-trivial procedure, the necessary constructions on the level of the supersymmetry algebra are very easy. We'll need to specify a supercharge Q (i.e. a fermionic element of the supersymmetry algebra) which satisfies $[Q,Q] = 2Q^2 = 0$, such that all the translations are Q exact. That is, for any translation T there exists a supercharge Q' so that $T = [Q, Q']$.

The upshot of specifying such data is that one can construct a *twist* with respect to Q of any theory acted on by the supersymmetry algebra. In this twist, we'll still have a supersymmetry algebra, but it will factor through the cohomology of the supersymmetry algebra with respect to the differential $[Q, -]$. Thus in particular all the translations are forced to act trivially, making the twisted theory *topological*.

So let's investigate what supercharges Q exist satisfying the required conditions. An easy way of doing this is to work in coordinates, so choose bases $\{e_1, e_2\}$, $\{f_1, f_2\}$ and $\{\alpha_1, \alpha_2\}$ for S_+, S_- and W respectively. The supercharges that square to zero are those of form $(\psi_+ \otimes (a_1\alpha_1 + a_2\alpha_2), \psi_- \otimes (b_1\alpha_1^* + b_2\alpha_2^*))$ where $a_1b_1 + a_2b_2 = 0$. This includes for instance all those supercharges concentrated in a single summand, $S_+ \otimes W$ or $S_- \otimes W^*$.

The image of the operator $\Gamma(Q, -)$ is the span of the images of projections onto the two summands. If Q is concentrated in $S_+ \otimes W$ say, the image of $\Gamma(Q, -)$ is the whole space of translations unless Q is contained in an affine subspace of form $\{\psi_+\} \otimes W$, in which case the image is the two-dimensional subspace $\{\psi_+\} \otimes S_- \subseteq \mathbb{C}^4$. Therefore all supercharges Q that square to zero are topological except for the measure zero subset of supercharges of this form, or those living in an affine subspace of form $\{\psi_-\} \otimes W^*$. For concreteness, we might concentrate on the fixed topological supercharge

$$Q_A = e_1 \otimes \alpha_1 + e_2 \otimes \alpha_2.$$

3 Supersymmetric Gauge Theories

I won't give a complete construction of $N = 2$ supersymmetric gauge theories, both because it would take too long, and because it would probably be unenlightening to see all the gory details. There are several ways of building a gauge theory with $N = 2$ supersymmetry, mainly by compactification of higher dimensional theories (for instance, if you like Kevin Costello's language for classical field theories you can read about his construction of pure $N = 2$ gauge theory on $\mathbb{R}^4$ in [Cos11]. I'll just describe the fields in $N = 2$ gauge theories, and talk about what sorts of theories exist. Then I'll go on to explain certain invariants associated to the theories, such as the moduli space of vacua. A general review of $N = 2$ gauge theories containing all this information is [Tac13].

Let G be a complex semisimple Lie group with Lie algebra $\mathfrak{g}$. For the rest of this section we'll discuss $N = 2$ supersymmetric gauge theories on $\mathbb{R}^4$ with gauge group G. Then at the end of this section we'll discuss topological twists of these gauge theories, which can be defined on any Riemannian 4-manifold X, not just $\mathbb{R}^4$.

3.1 Vector Multiplets

The word "multiplet" should be thought of as meaning a collection of fields interchanged by the action of the supersymmetries. For free supersymmetric theories, multiplets correspond to irreducible classical field theories: theories that do not factor as a non-trivial product of smaller theories. The most important sort of multiplet in a supersymmetric gauge theory is a *vector multiplet*, which is a multiplet including a gauge field, i.e. a connection.

An $N = 2$ vector multiplet consists of the following set of fields:

- A gauge field A, i.e. a connection on the trivial G-bundle over $\mathbb{R}^4$.
- A pair of $\mathfrak{g}$-valued spinors $(\psi, \widetilde{\psi}) \in \Omega^0(\mathbb{R}^4; \Pi S_+ \otimes W \otimes \mathfrak{g})$.
- A $\mathfrak{g}$ valued scalar $\Phi \in \Omega^0(\mathbb{R}^4; \mathfrak{g})$.

There are several ways of interpreting this data. Here are a few I know

1. An $N = 2$ supersymmetric field theory is a fortiori an $N = 1$ supersymmetric field theory, so we can view these components as coming from gluing together two different $N = 1$ multiplets. Gauge theories with $N = 1$ supersymmetry are comparitively easy to understand, for instance they have natural "superspace" interpretations, where one view a vector multiplet, for instance, as components of a gauge field on a supermanifold whose even piece is the spacetime we want. From this point of view, the gauge field and one of the spinors comprise an $N = 1$ vector multiplet, and the scalar and the other spinor comprise an $N = 1$ "chiral multiplet" (a spinor on superspace).

2. Another nice approach which works for several different 4d supersymmetric gauge theories is to view these components as coming from a *holomorphic* connection on a *super twistor space*. Super twistor space is a supermanifold projecting down to $\mathbb{R}^4$, whose fibres look like the total space of an odd vector bundle over $\mathbb{CP}^1$. The powerful fact (usually called the *Penrose-Ward correspondence*) is that super twistor space admits a natural complex structure, such that the classical field theory controlling holomorphic G-bundles is naturally isomorphic to supersymmetric gauge theory on $\mathbb{R}^4$ (up to technical modifications). A version of this theorem is proven by Boels, Mason and Skinner [BMS07].

3. A third approach, which again works for many different supersymmetric gauge theories, is to perform dimensional reduction from an $N = 1$ supersymmetric gauge theory in 10 dimensions, which is easy to define. When dimensionally reducing to 4 dimensions one actually obtains an $N = 4$ theory, but one can restrict to an $N = 2$ theory by discarding certain components, i.e. restricting to an $N = 2$ orbit. There's a neat description of this family of ideas in terms of the classification of normed division algebras in [ABD+13].

Each of these stories can be enriched to explain a natural action functional as well, but we won't try to do this: it won't be very enlightening for our purposes. We might however describe a general form for action functionals in $N = 2$ theories in terms of a function called the *prepotential*. This is a formula in an $N = 2$ superspace formalism. $N = 2$ superspace is the supermanifold $\mathbb{R}^4 \times \mathbb{C}^{0|4}$, where we denote the odd holomorphic coordinates by $\theta_a, \widetilde{\theta}_a$ for $a = 1, 2$. $N = 2$ superfields are then sections of vector bundles on superspace that are holomorphic in the odd directions. The superspace formulation of $N = 2$ gauge theory includes a G-connection $\mathcal{A}$ on superspace satisfying a certain constraint, and an $N = 2$ superfield Ψ such that $\overline{D}_{\mathcal{A}}\Psi = 0$ [DF99, 10.2].

Definition 3.1. A *prepotential* for an $N = 2$ theory in the superspace formalism is a $\mathbb{C}$-valued functional $\mathcal{F}$ on the space of global superfields. The associated action functional is the fermionic integral

$$\mathcal{L}(\Psi) = \operatorname{Im} \int d^2\theta d^2\widetilde{\theta}\mathcal{F}(\Phi).$$

Remark 3.2. Fermionic integration is actually a very simple thing. It's the linear map from functions on superspace to $\mathbb{C}$ uniquely determined by the conditions

$$\int d^2\theta \, (\theta_1\theta_2) = 1$$
$$\text{and} \int d^2\theta \frac{\partial f}{\partial \theta_i} = 0 \text{ for } i = 1, 2.$$

3.2 Hypermultiplets

Now, let's explain what sort of *matter fields* one can introduce into $N = 2$ gauge theories. By matter fields we mean multiplets containing a spinor and no fields of higher spin (whose Lagrangian will allow for a term giving the spinor a mass), these are called *hypermultiplets* and exist for any complex representation R of the gauge group equipped with a Hermitian form. They consist of the following component fields

- Two Weyl spinors $\lambda \in \Omega^0(\mathbb{R}^4; \Pi S_+ \otimes R)$ and $\widetilde{\lambda} \in \Omega^0(\mathbb{R}^4; \Pi S_+ \otimes \overline{R})$ valued in complex conjugate representations of G.
- Two scalars $\phi \in \Omega^0(\mathbb{R}^4; R)$ and $\widetilde{\phi} \in \Omega^0(\mathbb{R}^4; \overline{R})$, again valued in complex conjugate representations of G.

Again there are several interpretations of these component fields, for instance there's a version of each of the above interpretations for the vector multiplet that include this R-valued matter.

Examples 3.3. 1. The most common type of hypermultiplet considered in the literature is the case where $G = SU(V)$ and $R = V^{N_f}$. This is called $N = 2$ *Super QCD* with N_f flavours.

2. Another example that is often considered is *quiver gauge theory*. This is the case where we fix a quiver Γ and a vector space V_v for each vertex v, let $G = \prod_v \mathrm{SU}(V_v)$, and let $R = \bigoplus_e V_{h(e)} \otimes \overline{V}_{t(e)}$, where $h(e)$ and $t(e)$ are the head and tail vertices of an edge e.

3.3 Twisting

The idea of *twisting* a supersymmetric field theory is approximately the following. Given a topological supercharge (or more generally, any supercharge Q satisfying $Q^2 = 0$) one takes invariants of the classical theory with respect to the supergroup $\Pi\mathbb{C}$ generated by Q. So for instance, one restricts attention to Q-invariant classical observables, and treats all Q-exact observables as trivial. If Q is a topological supercharge then the resulting twisted theory is *topological*.

Let's describe the twist of the $N = 2$ theory with respect to the supercharge Q_A described in the previous section. The moduli space of solutions to the equations of motion on an open set U (for a pure gauge theory, i.e. without a hypermultiplet) is

$$\mathrm{EOM}(U) = T^*[-1]T^{1,0}[1](\mathcal{M}_{\mathrm{Inst}}(U))$$

where $\mathcal{M}_{\mathrm{Inst}}(U)$ is the moduli space of instantons over U modulo gauge. One can compute this at the perturbative level, i.e. the level of the tangent complex to the moduli space near a point, by taking explicit (derived) invariants with respect to the action of the topological supercharge. Here we're using a natural complex structure on the moduli space of instantons to define the holomorphic tangent bundle.

Once one has made this calculation, one can use it to *define* the twisted theory on a general Riemannian 4-manifold X, not necessarily $\mathbb{R}^4$. That means that from the starting point of $N = 2$ supersymmetric gauge theory we've produced a natural classical field theory on any Riemannian 4-manifold describing instantons, and thus the beginning of a story that allows us to recover Donaldson invariants from supersymmetric field theory.

What if we include a hypermultiplet? Well, for any representation V of G, there's a natural fibre bundle over $\mathcal{M}_{\mathrm{Inst}}$ of solutions to the Dirac equation in the instanton background, in representation V. So the fibre over an instanton A is the space of spinors ψ in $\Omega^0(X; (S_+ \oplus S_-) \otimes V)$ satisfying the Dirac equation $\overline{\psi} \not{D}_A \psi = 0$. Note that we might need to specify a $\mathrm{Spin}^{\mathbb{C}}$ structure on X in order for this to be well-defined.

3.4 Moduli of Vacua

The analysis of the moduli space of vacua in $N = 2$ theories is an important topic that we won't have time to do justice to here. At most I can give an idea of what the moduli space of vacua is, and what sorts of properties it has. To begin, here's an approximate definition.

Definition 3.4. The *scalar potential* in a supersymmetric field theory is the term in the action which is a polynomial in the scalar components of the multiplets (i.e. not involving any derivatives).

So in our $N = 2$ theories this will be a polynomial in Φ, ϕ and $\widetilde{\phi}$. Concretely it looks like (up to some scalar factors that won't be important for our purposes today)

$$V(\Phi, \phi, \widetilde{\phi}) = \mathrm{Tr}([\Phi^\dagger, \Phi]^2) + (h(\phi, \phi) - h(\widetilde{\phi}, \widetilde{\phi}))^2 + h(\widetilde{\phi}, (\Phi + M)\phi) + h(\phi, (\Phi + M)\widetilde{\phi})$$

where $h(-, -)$ is the Hermitian pairing on R, and where M is an element of $\mathrm{End}(R)$ which we think of as a matrix of masses. For convenience let's set $M = 0$, so the hypermultiplets are massless. This will make the moduli space of vacua better behaved.

Definition 3.5. The moduli space of (classical) *supersymmetric vacua* in a supersymmetric field theory is the space of supersymmetry invariant solutions to the equations of motion which are global minima of the scalar potential.

Let's describe some properties of the moduli space of vacua in an $N = 2$ theory.

- There's a subspace of the moduli space of vacua in which the scalars ϕ and $\widetilde{\phi}$ in the hypermultiplet both equal zero. This is called the *Coulomb branch* of the moduli space. We can see from our expression for V that on the Coulomb branch $\mathrm{Tr}([\Phi^\dagger, \Phi]^2)$ is minimised, i.e. $\mathrm{Tr}([\Phi^\dagger, \Phi]^2) = 0$, because this term of the potential is non-negative and attains zero at $\Phi = 0$. This holds if Φ is diagonal, so the Coulomb branch certainly contains a copy of the Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$.
- There's another subspace of the moduli space of vacua in which the scalar Φ in the vector multiplet is zero. This is called the *Higgs branch* of the moduli space. On the Higgs branch $(h(\phi, \phi) - h(\widetilde{\phi}, \widetilde{\phi}))^2$ is zero, so $h(\phi, \phi) = h(\widetilde{\phi}, \widetilde{\phi})$. The Higgs and Coulomb branches meet one another transversely at the trivial solution to the equations of motion, because supersymmetry invariance ensures that if the scalar fields vanish then so does everything else.
- There are projection maps down to the Higgs and Coulomb branches by setting the appropriate scalars to zero.
- In general, the Coulomb branch naturally has the structure of a Kähler manifold, and is the base of a physically interesting integrable system (the Hitchin base is an example of the Coulomb branch in an $N = 2$ theory).
- In general, the Higgs branch naturally has the structure of a hyperkähler manifold.

References

[ABD+13] Alexandros Anastasiou, Leron Borsten, Michael Duff, Leo Hughes, and Silvia Nagy. Super Yang-Mills, division algebras and triality. *arXiv preprint arXiv:1309.0546*, 2013.

[AGT10] Luis F Alday, Davide Gaiotto, and Yuji Tachikawa. Liouville correlation functions from four-dimensional gauge theories. *Letters in Mathematical Physics*, 91(2):167–197, 2010.

[BMS07] Rutger Boels, Lionel Mason, and David Skinner. Supersymmetric gauge theories in twistor space. *Journal of High Energy Physics*, 2007(02):014, 2007.

[Cos11] Kevin Costello. Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. *arXiv preprint arXiv:1111.4234*, 2011.

[DF99] Pierre Deligne and Daniel Freed. Supersolutions. *Quantum fields and strings: a course for mathematicians*, Volume 1:227–355, 1999.

[Don83] Simon Donaldson. An application of gauge theory to four dimensional topology. *Journal of Differential Geometry*, 18(2):279–315, 1983.

[HŁS75] Rudolf Haag, Jan Łopuszański, and Martin Sohnius. All possible generators of supersymmetries of the S-matrix. *Nuclear Physics B*, 88(2):257–274, 1975.

[Nek03] Nikita Nekrasov. Seiberg-Witten prepotential from instanton counting. *Advances in Theoretical and Mathematical Physics*, 7(5):831–864, 2003.

[Rod13] Robert Rodger. A pedagogical introduction to the AGT conjecture. Master's thesis, Universiteit Utrecht, 2013.

[SW94] Nathan Seiberg and Edward Witten. Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nuclear Physics B*, 431(3):484–550, 1994.

[Tac13] Yuji Tachikawa. $N = 2$ supersymmetric dynamics for pedestrians. *arXiv preprint arXiv:1312.2684*, 2013.

[Wit88] Edward Witten. Topological quantum field theory. *Communications in Mathematical Physics*, 117(3):353–386, 1988.