Gauge Symmetry Breaking via Derived Geometry

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Plan for Today

- We'll start with a perspective on classical field theory via "derived geometry".
- How to think about symmetry breaking from this perspective. Main example: Yang–Mills–Higgs theory.
- More exotic examples: "twisted supergravity".

I'm going to discuss joint work with Owen Gwilliam (`arxiv.org/abs/2008.03599`).

Symmetry Breaking in Physics

Main goal, understand the following idea.

In a classical field theory, there is a moduli space of "vacua". If one moves away from special "unstable" vacua the theory changes.

Typically it becomes less symmetrical, and massless particles can "become massive". Key example in physics: the gauge symmetry breaking studied in the 1960s by Higgs, Anderson, Englert–Brout, Guralnik–Hagen–Kibble.

Classical Field Theory, Informally

Starting point to classical field theory:

- A space Φ of fields (e.g. space of smooth sections of a vector bundle).
- An action functional $S\colon \Phi \to \mathbb{R}$.

Classical field theory studies the critical locus of S. One way of describing this geometrically is as an intersection.

$$\mathrm{Crit}(S) = \mathrm{Graph}(\mathrm{d}S) \cap_{T^*\Phi} \Phi.$$

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This intersection will not be transverse. We will replace the critical locus by a more refined notion from algebraic geometry.

Suppose $X_1 \to Y \leftarrow X_2$ are affine algebraic varieties, $X_i = \mathrm{Spec}\, R_i$, $Y = \mathrm{Spec}\, S$. The fiber product is

$$X_1 \times_Y X_2 = \mathrm{Spec}(R_1 \otimes_S R_2).$$

Definition

The derived fiber product of X_1 and X_2 over Y is

$$X_1 \times_Y^h X_2 = \mathrm{Spec}(R_1 \otimes_S^{\mathbb{L}} R_2).$$

The derived fiber product "knows about how X_1 and X_2 intersect", e.g. the difference between transverse and non-transverse intersections.

We would like to try to study

$$\mathrm{dCrit}(S) = \mathrm{Graph}(\mathrm{d}S) \times^h_{T^*\Phi} \Phi,$$

but making this precise is tricky: Φ is usually some kind of infinite-dimensional topological vector space. We'll do something a bit easier. If we choose a point $x \in \mathrm{Crit}(S)$, we can study the formal neighbourhood of x in $\mathrm{dCrit}(S)$. We can describe this using linear algebra!

Symmetry breaking will come in when we vary x. Because $\mathrm{dCrit}(S)$ is not smooth, formal neighbourhoods of different points can look quite different.

The BV Formalism

The BV formalism models the tangent space at x to $\mathrm{dCrit}(S)$. In the derived setting the tangent space is a cochain complex!

Example: Yang–Mills–Higgs theory

Fix a Riemannian manifold X, a semisimple Lie algebra $\mathfrak{g}$ and a representation R. We'll also fix a $\mathfrak{g}$-invariant potential functional

$$V \colon R \to \mathbb{R},$$

the usual example is $V(r) = \frac{1}{4}(|r|^2 - m^2)^2$. I'll write down a model for the formal neighbourhood of the trivial solution to the Yang–Mills–Higgs equations.

$$\begin{aligned} \mathrm{d}_A * \mathrm{d}_A(A) &= \langle \phi, \mathrm{d}_A \phi \rangle \\ \mathrm{d}_A * \mathrm{d}_A(\phi) &= -\nabla V(\phi). \end{aligned}$$

Roughly speaking, the classical BV complex looks like $\Phi \xrightarrow{D} \Phi^*$, where the operator D is given by the second Taylor coefficient of the action functional. Let's explain the Yang–Mills–Higgs example piece by piece.

I won't explain the general algorithm here, for more details see `sites.google.com/view/physical-mathematics-of-qfts/`

Unpacking the Yang–Mills–Higgs Example

$$\Omega^0(X, \mathfrak{g}) \xrightarrow{\mathrm{d}} \Omega^1(X, \mathfrak{g}) \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^3(X, \mathfrak{g}) \xrightarrow{\mathrm{d}} \Omega^4(X, \mathfrak{g})$$

$$\Omega^0(X; R) \xrightarrow{\mathrm{d}*\mathrm{d}-m^2*} \Omega^4(X; R)$$

Symmetry Breaking

Remember that we chose a classical solution x, and worked in a formal neighbourhood around it. What if we vary x?

For example, in Yang–Mills–Higgs theory, consider a constant function whose value ϕ_0 is a critical point of V. So this means $|\phi_0|^2 = m^2$, or $\phi_0 = 0$. The Taylor coefficients of the action functional change, so the differential in the BV complex changes!

Suppose $\phi_0 \neq 0$ is a critical point of V. The Taylor expansion of the action functional looks like

$$\frac{1}{2}(-|\mathrm{d}A\|^2 + |\mathrm{d}\phi|^2 + |\phi_0|^2|A|^2 - m^2|\phi|^2 + \langle \phi_0, \phi \rangle^2) + \cdots .$$

The differential on the classical BV complex at ϕ_0 deforms correspondingly:

$$\begin{array}{ccccccc} \Omega^0(X,\mathfrak{g}) & \xrightarrow{\mathrm{d}} & \Omega^1(X,\mathfrak{g}) & \xrightarrow{\mathrm{d}*\mathrm{d}+|\phi_0|^2*} & \Omega^3(X,\mathfrak{g}) & \xrightarrow{\mathrm{d}} & \Omega^4(X,\mathfrak{g}) \\ & \searrow & & \nearrow \searrow & & \nearrow & \\ & & \Omega^0(X;R) & \xrightarrow{\mathrm{d}*\mathrm{d}-m^2 p} & \Omega^4(X;R) & & \end{array}$$

where the diagonal arrows are proportional to ϕ_0 and $p\colon R \to R$ is the projection onto $(\mathbb{R}\phi_0)^\perp \subseteq R$.

Suppose $\mathfrak{h} \subseteq \mathfrak{g}$ is the stabilizer of ϕ_0, and decompose $\mathfrak{g}$ into $\mathfrak{h} \oplus \mathfrak{h}^\perp$. Likewise decompose R into $\mathbb{R}\phi_0 \oplus (\mathbb{R}\phi_0)^\perp$.

Theorem

There is a deformation retract from the classical BV complex near ϕ_0 onto the following complex.

$$\Omega^0(X, \mathfrak{h}) \xrightarrow{\mathrm{d}} \Omega^1(X, \mathfrak{h}) \xrightarrow{\mathrm{d}*\mathrm{d}} \Omega^3(X, \mathfrak{h}) \xrightarrow{\mathrm{d}} \Omega^4(X, \mathfrak{h})$$

$$\Omega^1(X; \mathfrak{h}^\perp) \xrightarrow{\mathrm{d}*\mathrm{d}-m^2*} \Omega^3(X; \mathfrak{h}^\perp)$$

$$\Omega^0(X; (\mathbb{R}\phi_0)^\perp) \xrightarrow{\mathrm{d}*\mathrm{d}-m^2*} \Omega^4(X; (\mathbb{R}\phi_0)^\perp).$$

Electroweak Theory

For a standard example, suppose $G = \mathrm{SU}(2) \oplus \mathrm{U}(1)$, and suppose R is the two dimensional defining representation of $\mathrm{SU}(2)$, with $\mathrm{U}(1)$ acting with weight 1.

One chooses a non-zero critical point ϕ_0 to perturb around so that the symmetry is broken to $\mathfrak{h} = \mathrm{u}(1) \subseteq \mathfrak{su}(2) \oplus \mathrm{u}(1)$ (embedded via $x \mapsto (h/2 \cdot x, x)$).

So $\mathfrak{h}^\perp$ is 3-dimensional (spanned by the massive W_+, W_- and Z bosons) and $(\mathbb{R}\phi_0)^\perp$ is 1-dimensional (spanned by the Higgs boson).

Further Examples

More exotic examples arise when we study supergravity theory. Can model supergravity using gauge theories for $\mathbb{Z}/2\mathbb{Z}$-graded extensions of the Poincaré group $\mathrm{SO}(4) \ltimes \mathbb{R}^4$.

Idea: One of the fields $q_\pm$ in this theory lives in $\Omega^0(\mathbb{R}^4; S_\pm)$ where $S_\pm$ are the semispin representations of $\mathfrak{so}(4)$. This fields lives in $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-bidegree $(-1, 1)$: "bosonic ghost". Study symmetry breaking where this field $q_\pm$ is given a non-zero value.

One can show that this theory at non-zero $q_\pm$ is holomorphic. In particular this means it is easier to quantize.

Conjecture (E–Williams)

This "twisted supergravity theory" is equivalent to a theory modelling holomorphic symplectic vector fields, whose quantization we constructed in arxiv.org/abs/2008.02302.

Thanks for listening!