# $\Omega$-Background and the Nekrasov Partition Function

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## 1 Introduction

In this talk we'll aim to make sense of the Nekrasov partition function (as introduced in Nekrasov's ICM address [Nek03]) as the partition function in a twisted deformed $N = 2$ gauge theory. While making the deformation is unnecessary – without it we're just talking about the partition function in Donaldson-Witten theory – deforming the theory eliminates IR divergences, allowing a direct mathematical definition of the partition function as the pushforward of an equivariant cohomology class on instanton moduli space. We'll conclude by stating the result of Nekrasov and Okounkov [NO06] relating the Nekrasov partition function to the Seiberg-Witten prepotential. As well as the papers of Nekrasov and Nekrasov-Okounkov, a good background reference is the masters thesis of Rodger [Rod13].

In what follows we'll discuss $N = 2$ super Yang-Mills theory on $\mathbb{R}^4$ with gauge group $G$ and no matter fields. The story below can be generalised to include matter living in hypermultiplets corresponding to various representations, but we won't discuss this today.

There are a few points that I haven't managed to figure out while preparing these notes. I've indicated them as **questions**, and I hope we can address them during our discussions.

## 2 $\Omega$-Background and $\Omega$-Twists

### 2.1 The $\Omega$-Deformed Theory

One of the ways we can produce $N = 2$ supersymmetric gauge theory in four-dimensions is by *dimensional reduction* from $N = 1$ supersymmetric gauge theory in six dimensions. I don't know the best way of defining dimensional reduction mathematically, but physically the idea is the following. There's a very well-defined procedure called *compactification*, which produces a classical field theory on a manifold $X$ from a theory on a product $X \times Y$ by pushing forward along the first projection map (that is, the classical states on $U \subseteq X$ after pushing forward are the classical states on $U \times Y \subseteq X \times Y$ in the original theory). The *dimensional reduction* is the theory obtained as the limit of the compactified theories as the diameter of $Y$ under its pseudo-Riemannian metric shrinks to zero.

In this language, the $N = 2$ theory on $\mathbb{R}^4$ is obtained by dimensional reduction from the $N = 1$ theory on $\mathbb{R}^4 \times T^2$ with its flat metric. The $\Omega$-*deformation* is a different dimensionally reduced theory obtained by first deforming the metric on $\mathbb{R}^4 \times T^2$.

**Definition 2.1.** The $\Omega$-deformation of $N = 2$ super Yang-Mills on $\mathbb{R}^4$ is the dimensional reduction of $N = 1$ super Yang-Mills on $\mathbb{R}^4 \times T^2$ with Riemannian metric

$$ds^2 = Adzd\overline{z} + \sum_{i=1}^{4}(dx^i + \Omega^i_j x^j dz + \overline{\Omega}^i_j x^j d\overline{z})^2$$

where $x^1, \dots, x^4$ are coordinates on $\mathbb{R}^4$, $z, \overline{z}$ are complex conjugate coordinates on $T^2$, and $A$ is an area parameter which we'll send to zero. In this basis, $\Omega^i_j$ and $\overline{\Omega}^i_j$ are given by the matrix

$$\Omega^i_j = \begin{pmatrix} 0 & \varepsilon_1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \end{pmatrix}$$

and its complex conjugate respectively, where $\varepsilon_1$ and $\varepsilon_2$ are real parameters.

A more geometric way of thinking of the Ω-deformation is to think of $\mathbb{R}^4 \times T^2$ as a trivial vector bundle over $T^2$. One can replace this by the vector bundle $(\mathbb{R}^4 \times \mathbb{R}^2)/\mathbb{Z}^2$ where the generators of $\mathbb{Z}^2$ act on $\mathbb{R}^4$ by $\Re\mathrm{e}\,\Omega$ and $\Im\mathrm{m}\,\Omega$ respectively. A priori there's no reason for this deformed theory to retain any supersymmetry, but for our judicious choice of Ω one can check that we still have a well-defined action of a deformation of the *Donaldson-Witten supercharge*. We'll describe this now.

## 2.2 Topological Twists

Let's recall the definition of the $N = 2$ supersymmetry algebra, in order to set up some notation.

**Definition 2.2.** The (complexified) $N = 2$ supersymmetry algebra is

$$(\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_- \oplus \mathfrak{gl}(2;\mathbb{C})_R \oplus \mathbb{C}^4) \oplus \Pi(S_+ \otimes W \oplus S_- \otimes W^*)$$

where $W$ is a two-dimensional complex vector space, and where the brackets are given by the obvious internal brackets on the factors, the action of $\mathfrak{sl}(2;\mathbb{C})_\pm$ on $S_\pm$, the action of $\mathfrak{gl}(2;\mathbb{C})_R$ on $W$ and $W^*$, and the Γ-pairing $S_+ \otimes S_- \to \mathbb{C}^4$.

Choose bases $\{\alpha_1, \alpha_2\}$ and $\{\alpha_1^\vee, \alpha_2^\vee\}$, and a basis $\{e_1, e_2\}$ for $W$. We can define a supercharge which is of particular historical interest for $N = 2$ theories: the supercharge whose twist was first studied by Witten in his approach to Donaldson theory [Wit88].

**Definition 2.3.** The *Donaldson-Witten supercharge* is the square-zero fermionic element $Q_{\mathrm{DW}} = \alpha_1 \otimes e_1 - \alpha_2 \otimes e_2$ of the supersymmetry algebra. This has the property that $\Gamma(Q_{\mathrm{DW}}, -)\colon S_+ \otimes W \oplus S_- \otimes W^* \to \mathbb{C}^4$ is surjective, so this is a *topological* supercharge.

It will also be useful to fix the notation $(Q_1, Q_2, Q_3, Q_4) = (\alpha_1^\vee \otimes e_1^*, \alpha_2^\vee \otimes e_1^*, \alpha_1^\vee \otimes e_2^*, \alpha_2^\vee \otimes e_2^*)$.

An $N = 2$ superysmmetric classical field theories admits an action of the supersymmetry algebra, in particular an action of the supergroup $\Pi\mathbb{C}$ generated by $Q_{\mathrm{DW}}$. We'll describe, following Nekrasov, a family of fermionic square-zero symmetries deforming this action.

**Definition 2.4.** Let Ω be an element of $\mathfrak{so}(4)$. We define a fermionic symmetry $Q_\Omega$ deforming $Q_{\mathrm{DW}}$ by

$$Q_\Omega = Q_{\mathrm{DW}} + \Omega^i_j x^j Q_i.$$

where $\{x^i\}$ is a basis for $\mathbb{R}^4$, acting on fields in the field theory by dilation in the appropriate coordinate direction.

We'll particularly be interested in the family of symmetries $Q_{\varepsilon_1\varepsilon_2}$ associated to elements Ω of form

$$\Omega = \begin{pmatrix} 0 & \varepsilon_1 & 0 & 0 \\ -\varepsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & -\varepsilon_2 & 0 \end{pmatrix},$$

where $\varepsilon_1$ and $\varepsilon_2$ are real numbers. This provides a two-parameter deformation of the action of the Donaldson-Witten supercharge.

**Proposition 2.5.** The symmetry $Q_{\varepsilon_1\varepsilon_2}$ squares to zero.

*Proof.* The Donaldson-Witten supercharge squares to zero, as does each $Q_i$, and $[Q_i, Q_j] = 0$ for all $i$ and $j$, so it suffices to observe that $[Q_{\mathrm{DW}}, \Omega^i_j x^j Q_i] = 0$. We compute

$$\begin{aligned}[Q_{\mathrm{DW}}, \Omega^i_j x^j Q_i] &= \Omega^i_j x^j [Q_{\mathrm{DW}}, Q_i] \\ &= \Omega^i_j x^j x^i \\ &= (x_1, x_2, x_3, x_4) \cdot (\varepsilon_1 x_2, -\varepsilon_1 x_1, \varepsilon_2 x_4, -\varepsilon_2 x_3)^T \\ &= 0\end{aligned}$$

as required. □

**Claim.** The $\Omega$-deformed symmetry $Q_{\varepsilon_1\varepsilon_2}$ acts on the $\Omega$-deformed $N = 2$ gauge theory as a square-zero symmetry. Thus, in particular we can *twist* by the symmetry $Q_{\varepsilon_1\varepsilon_2}$, yielding a a family of twisted theories deforming Donaldson-Witten gauge theory.

### 2.3 Observables

Having defined the deformed supercharges $Q_{\varepsilon_1\varepsilon_2}$, and the deformed theory on which they act, we'll describe certain interesting observables in the $N = 2$ theory which are invariant under the action of this supercharge, i.e. observables in the twisted deformed theory. The natural gauge invariant observables which one considers in an $N = 2$ gauge theory are those of form

$$\mathcal{O}_{P,\omega}(A, \psi, \widetilde{\psi}, \phi) = \int_{\mathbb{R}^4} \omega \wedge P_*(F_A + \psi + \widetilde{\psi} + \phi)$$

where $\omega \in \Omega^\bullet(\mathbb{R}^4)$ is a differential form (not necessarily homogeneous), and $P$ is a $G$-invariant polynomial on $\mathfrak{g}$ under which we can push forward a $\mathfrak{g}$-valued form. Strictly speaking, since $\mathbb{R}^4$ is non-compact, we must require that $\omega$ is compactly supported, or at least goes to zero sufficiently fast near infinity, in order to ensure that this integral is always well-defined. Of course, the exact same problem often occurs with the action functional itself on non-compact manifolds, and generally vanishes in the formal manipulation involved in evaluation of the path integral.

## 3 The Nekrasov Partition Function

At this point we've introduced enough background to actually define the Nekrasov partition function. After doing this we'll use what we've learned about the twist, as well as material we've learned in previous talks, to extract a more mathematically meaningful definition.

**Definition 3.1.** The *Nekrasov partition function* $Z(a; \varepsilon_1, \varepsilon_2)$ is the partition function in the $Q_{\varepsilon_1\varepsilon_2}$ twisted $N = 2$ gauge theory, evaluated in a vacuum state $a$.

We should take a moment to recall what this means exactly. The *partition function* in perturbation theory is the expectation value of the trivial observable, $\langle 0|1|0\rangle$. While this is not well-defined – regularization usually allows us to arbitrarily rescale the partition function – if one has a family of field theories one can think of the partition function as a flat line bundle over the parameter space which can be trivialised to obtain a function. What's more, in perturbation theory one has to fix a choice of vacuum state $a = |0\rangle$ to perturb around. As we learned in a previous talk, in pure $N = 2$ gauge theory the moduli space of vacua is isomorphic to the Cartan subalgebra $\mathfrak{h}$ of the gauge group.

Since we're working in the $Q_{\varepsilon_1\varepsilon_2}$ twisted theory, we can replace the observable 1 by any observable of form $1+Q_{\varepsilon_1\varepsilon_2}\mathcal{O}$. We'll use this fact to give an equivalent definition of the Nekrasov partition function. Choose an identification

$\mathbb{R}^4 \cong \mathbb{C}^2$, and let

$$\mathcal{O}_{\varepsilon_1\varepsilon_2}(A,\psi,\widetilde{\psi},\phi) = \exp\left(\frac{1}{(2\pi i)^2}\int_{\mathbb{C}^2}(\omega + H_{\varepsilon_1\varepsilon_1})\wedge \mathrm{Tr}(F_A\phi + \frac{1}{2}\psi\psi + F_A\wedge F_A)\right)$$

where $\omega = dz_1\wedge d\overline{z}_1 + dz_2\wedge d\overline{z}_2$ is the standard symplectic form, and $H_{\varepsilon_1\varepsilon_1} = \varepsilon_1(|z_1|^2)+\varepsilon_2(|z_2|^2)$. Nekrasov [Nek03] argues that the observable $\mathcal{P}_{\varepsilon_1\varepsilon_2} = \int_{\mathbb{C}^2}((\omega + H_{\varepsilon_1\varepsilon_1})\wedge \mathrm{Tr}(F_A\phi + \frac{1}{2}\psi\psi + F_A\wedge F_A))$ is $Q_{\varepsilon_1\varepsilon_2}$-exact, thus so is its $n^{\text{th}}$ power for each $n$, so the exponential reduces to the constant term 1 alone in the $Q_{\varepsilon_1\varepsilon_2}$ cohomology. He claims that a pre-image is provided by

$$\mathcal{P}_{\varepsilon_1\varepsilon_2} = Q_{\varepsilon_1\varepsilon_2}\int_{\mathbb{C}^2}\int_{\mathbb{C}^{0|2}} d^4\theta((z_1\overline{\theta}_1 - \overline{z}_1\theta_1)dz_1\wedge d\overline{z}_1 + (z_2\overline{\theta}_2 - \overline{z}_2\theta_2)dz_2\wedge d\overline{z}_2)\wedge \mathrm{Tr}(\Phi^2)$$

using the superspace formalism, where we've written $\mathrm{Tr}(\Phi^2)$ rather than $\mathrm{Tr}(F_A\phi + \frac{1}{2}\psi\psi + F_A\wedge F_A)$ for short, but I haven't managed to check this.

**Question.** This doesn't look quite right, it seems like the Berezin integral should vanish. What is a correct preimage for the observable $\mathcal{P}_{\varepsilon_1\varepsilon_2}$ realising it as exact with respect to our deformed supercharge $Q_{\varepsilon_1\varepsilon_2}$?

To conclude this section, we'll state the main theorem relating the Nekrasov partition function to the Seiberg-Witten prepotential.

**Theorem 3.2** (Nekrasov-Okounkov [NO06])**.** The expression

$$\mathcal{F}(a) = \lim_{\varepsilon_1,\varepsilon_2\to 0}\varepsilon_1\varepsilon_2 \log Z(a;\varepsilon_1,\varepsilon_2)$$

coincides with the instanton part of the Seiberg-Witten prepotential.

Since – as we'll discuss in the next section – the Nekrasov partition function is practically computable, this gives the most powerful known method for computing the prepotential to all orders in the instanton number.

## 3.1 The Nekrasov Partition Function and Instantons

We haven't explained yet exactly what the Ω-deformation buys for us, so let's explain that now. The idea is that we can apply the technique of equivariant localization to the undeformed path integral, and the modification that one performs coincides with the Ω-deformation. A good explanation appears in [Rod13, 3.2.3]. Since the $Q_{\varepsilon_1\varepsilon_2}$ twisted theory is topological, the partition function is invariant under the renormalization group flow, so can be computed in the UV limit $\Lambda \to 0$, where the vacua dominate. In other words, our partition function is equal to an integral over the moduli space of instantons. This moduli space is non-compact, which we can deal with in two steps.

1. First, the moduli space of instantons has connected components indexed by the instanton number $k \in \mathbb{Z}$. We'll treat the path integral as a formal generating function, i.e. an integral of the form

$$\sum_{k\in\mathbb{Z}} q^k \int_{\mathcal{M}_k}\mathcal{O}$$

where $q$ is a formal parameter.

2. Secondly, while each component $\mathcal{M}_k$ is not compact, we can replace it with its Uhlenbeck compactification $\widetilde{\mathcal{M}_k}$ to obtain an integral over a compact space.

After making these modifications, we can give a well-defined mathematical definition of the partition function in question. First let's see what happens when we restrict the observables $\mathcal{O}_{\varepsilon_1\varepsilon_2}$ to the instanton states (without worrying too much about divergence issues). We find

$$\mathcal{O}_{\varepsilon_1\varepsilon_2}(A,0,0,a) = \exp\left(\frac{1}{(2\pi i)^2}\int_{\mathbb{C}^2}\omega\wedge \mathrm{Tr}(aF_A) + kH_{\varepsilon_1\varepsilon_2}\mathrm{dvol}\right)$$

where $k$ is the instanton number of $A$. The partition function in the UV limit then becomes

$$Z(a;\varepsilon_1,\varepsilon_2)=\sum_{q\in\mathbb{Z}}q^k\int_{\widetilde{\mathcal{M}_k}}\exp\left(\frac{1}{(2\pi i)^2}\int_{\mathbb{C}^2}\omega\wedge\mathrm{Tr}(aF_A)+2kH_{\varepsilon_1\varepsilon_2}\mathrm{dvol}\right)dA$$

where the extra factor of $kH_{\varepsilon_1\varepsilon_2}$ came from evaluating the action at the instanton states.

**Definition 3.3.** The *Nekrasov partition function* is the path integral

$$Z(a;\varepsilon_1,\varepsilon_2)=\sum_{k\in\mathbb{Z}}q^k\int_{\widetilde{\mathcal{M}_k}}\exp(\omega+\mu_G(a)+\mu_{T^2}(\varepsilon_1,\varepsilon_2)),$$

where $\omega$ is the symplectic form on the moduli space of instantons, and $\mu_G$ and $\mu_{T^2}$ are the moment maps associated to the $G$ action by rotating instantons at infinity, and the restriction of the SO(4) action on spacetime to its maximal torus.

One should be able to show the equivalence of this definition with the one above using the ADHM construction for instanton moduli space and some suitable regularization procedure.

**Question.** 1. How exactly does this work?

2. Why was it necessary to introduce the $\Omega$-deformation in order for this procedure to succeed? Or was it important for another reason?

Once one has got this far, we can apply the technique of localization to actually evaluate the integral.

**Theorem 3.4** (Duistermaat-Heckman Localization)**.** If $(X,\omega)$ is a compact symplectic $2n$-manifold with a Hamiltonian action of a torus $T$, then there is an equality

$$\int_X\exp(\omega+\mu_T(\xi))=\sum_{f\colon V_\xi(f)=0}\frac{\exp(\mu_T(\xi)(f))}{\prod_{i=1}^n w_i(\xi)(f)}$$

where $\xi\in\mathfrak{t}$, the Lie algebra of $T$, $V_\xi$ is the Hamiltonian vector field associated to $\xi$, and $w_i(\xi)(f)$ are the weights of the $T$ action on the tangent space at the fixed point associated to $f$.

In particular this theorem can be applied when $X=\widetilde{\mathcal{M}_k}$ and $T$ is the product of the maximal torus in $G$ and the maximal torus $T^2$ in SO(4) to compute the Nekrasov partition function. One finds for $G=\mathrm{SU}(N)$ that the instanton number $k$ piece of the partition function can be written as a sum over $N$-tuples of *partitions* of $k$. The formula can be found in [NO06, 3.1], and I won't reproduce it here in general, but it admits a nice simplification when $\varepsilon_1+\varepsilon_2=0$ which I will reproduce. One finds, up to a regulator factor,

$$Z(a,\varepsilon,-\varepsilon)=\sum_{\boldsymbol{k}}q^k\prod_{(l,i)\neq(m,j)}\frac{a_{lm}+\varepsilon(k_{l,i}-k_{m,j}+j-i)}{a_{lm}+\varepsilon(j-i)}$$

where the sum is over $N$-tuples $(\boldsymbol{k}_1,\ldots\boldsymbol{k}_N)$ of partitions of $k$ whose entries we denote in non-increasing order by $k_{i,j}$.

One answer to the question "what role does the $\Omega$-deformation play" might be provided by the AGT conjecture. This is a relationship between the Nekrasov partition function with matter living in an appropriate representation, and the function of conformal blocks in a 2d conformal field theory on the thrice punctured projective line. The conformal field theory that arises involves a potential term of form $e^{b\phi}$, where $\phi$ is a complex scalar field and $b$ is a free parameter. The appropriate values of the parameters $\varepsilon_1,\varepsilon_2$ in this correspondence are not zero, but instead should be set to $(b,1/b)$.

# References

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