Supersymmetric Lagrangians

Chris Elliott

February 6, 2013

1 Setup

Let's recall some notions from classical Lagrangian field theory. Let M be an oriented supermanifold: our *spacetime*, of dimension $n|d$. For our purposes, we will most often think about the case where M is either *Minkowski space* $\check{M}^n = \mathbb{R}^{1,n-1}$, or *Super Minkowski space*

$$M = \check{M}^n \times \Pi S$$

where S is a representation of $\mathrm{Spin}(1, n-1)$.

Definition 1.1. Let $E \to M$ be a smooth (super) fibre bundle. The associated space of *fields* is the space of smooth sections $\phi M \to E$. A *Lagrangian density* on $\mathcal{F}$ is a function $\mathcal{L} \colon \mathcal{F} \to \mathrm{Dens}(M)$ – where $\mathrm{Dens}(M)$ denotes the bundle of densities on M – that satisfies a *locality* condition.

Precisely, we form the pullback bundle

$$\begin{tikzcd} \mathrm{Dens}(M)_{\mathcal{F}} \arrow[r] \arrow[d] & \mathrm{Dens}(M) \arrow[d] \\ M \times \mathcal{F} \arrow[r, "\pi_1"'] & M \end{tikzcd}$$

and require $\mathcal{L}$ to be a section of this bundle such that, for some k, for all $m \in M$, $\mathcal{L}(m, \phi)$ only depends on the first k derivatives of ϕ. We can phrase this in terms of factoring through the k^{th} jet bundle of E.

As usual, we define the *action* functional to be the integral

$$S(\phi) = \int_M \mathcal{L}(\phi).$$

A Lagrangian system generally includes additional information, namely a choice of variational 1-form γ. In general I won't use this data, but I should mention when it might play a role.

1.1 Symmetries

In this talk, we will be discussing Lagrangian systems with certain kinds of symmetry called *supersymmetry.* As such, it'll be important to understand what it actually means to be a symmetry of such a system. Heuristically, a symmetry is just an automorphism of the space of fields that preserves the Lagrangian. It'll be easier for us to work with *infinitesimal* symmetries: symmetries will form a kind of Lie group, and infinitesmial symmetries will be elements of the Lie algebra: tangent vectors at the identity symmetry.

Definition 1.2. An *infinitesimal symmetry* of the Lagrangian system $(\mathcal{F}, \mathcal{L})$ is a vector field $\xi \in \mathrm{Vect}(M \times \mathcal{F})$, that is *local*, and *preserves the action.* More precisely

1. Locality means, analogously to locality for a Lagrangian, we require that there is some number k, such that ξ_ϕ only depends on at most k derivatives of ϕ for any $m \in M$.

2. Preserving the action means that
$$\mathrm{Lie}_\xi \mathcal{L} = d\alpha_\xi$$
for some (twisted) $n-1$-form α_ξ on $\mathcal{F}$.

Again, one could phrase the locality condition in terms of the k^{th} jet bundle of E.

Why does this exactness condition correspond to preserving the action? This is a consequence of the usual variation description of the space of classical solutions of a Lagrangian system. Indeed, first remember what the Lie derivative means:
$$\mathrm{Lie}_\xi \mathcal{L}(\phi) = \left.\frac{d}{dt}\right|_{t=0} \mathcal{L}(\exp(t\xi)^*\phi).$$
That is, the Lie derivative describes the infinitesimal variation of the function $\mathcal{L}$ in the direction of the tangent vector ξ. So this Lie derivative vanishes if the Lagrangian density is constant along such infinitesimal variations. Of course, we don't need to preserve the density, only its integral: the action. This integral is preserved if making such an infinitesimal modification changes the Lagrangian density by an exact term by Stokes's theorem.

We should say more about the calculus of variations, and what we mean by a variation (speaking heuristically only). Recall the idea: if we have an action $S = \int \mathcal{L}$ that we want to extremise over a space of fields (e.g. most classically a path space), we can consider the behaviour of this functional in a small neighbourhood of a given field ϕ. That is, we look at small variations $\phi + \delta\phi$ within this neighbourhood, and ask when
$$\int \mathcal{L}(\phi + \delta\phi) - \mathcal{L}(\phi) = 0$$
for all such variations. In the limit as the size of the neighbourhood goes to zero, this corresponds to computing the linear term in a Taylor expansion for $\mathcal{L}$. We call such extremal values of ϕ *classical solutions* of the Lagrangian system, and denote the space of such solutions by $\mathcal{M}$.

2 Supersymmetry

2.1 Translation and Supertranslation

For this section, suppose $M = (\check{M})^n \times \Pi S$ is super Minkowski space. On this space, we have an infinitesimal action of the *super translation algebra*
$$T = V \times \Pi S$$
where V is the even abelian Lie algebra $\mathbb{R}^{1,n}$, and the bracket comes from a choice of symmetric bilinear pairing $\Gamma\colon S \otimes S \to V$, or in coordinates
$$[Q_a, Q_b] = \Gamma^c_{ab} x_c$$
For x_i and Q_j bases for the even and odd parts of the vector space T. Such a pairing always exists, and in the case where S is irreducible is unique up to a scalar

If $(\mathcal{F}, \mathcal{L})$ defines a classical field theory on $(\check{M})^n$, we can extend the action of the translation algebra V on super Minkowski space to an action on $(\check{M})^n \times \mathcal{F}$. Indeed, as a Lie group, $v \in \mathbb{R}^n$ acts on fields by precomposition
$$(\check{M})^n \stackrel{+v}{\to} (\check{M})^n \to E,$$
and we can differentiate this action to produce an action of the Lie algebra. That is, an element $v \in V$ acts on fields by
$$v \cdot \phi = \left.\frac{d}{dt}\right|_{t=0} \exp(tv)^*\phi$$
which is just $\phi(x) \mapsto \phi(x-v)$. It is easy to see that this gives an action by symmetries, i.e. the map $V \hookrightarrow \mathrm{Vect}(\mathcal{F})$ lands in the subalgebra of symmetries. Certainly the action is local, and translation preserves the Lagrangian. This is the kind of symmetry we'd like to extend to a supergeometry setting.

2.2 Supersymmetric Lagrangians

Definition 2.1. A classical field theory $(\mathcal{F}, \mathcal{L})$ is *supersymmetric* if this action of translation extends to an action of the super translation algebra T by symmetries. .

If $n \equiv 1, 3, 4 \mod 4$, then there is a unique minimal irreducible spin representation S, so $T = V \times \Pi S^N$. We say there are N supersymmetries. If $n \equiv 2 \mod 4$, there are two minimal irresucible spin representations S^+, S^-, so $T = V \times \pi((S^+)^{N_1} \oplus (S^-)^{N_2})$. We say there are (N_1, N_2) supersymmetries.

Remark 2.2. We don't need the symmetries to be manifest here, so we should be careful with our computations.

Let's unpack this. For simplicity, suppose S is irreducible, i.e. that there is $N = 1$ supersymmetry. What do we need to do to define a supersymmetry? Well, let $Q_1, \ldots, Q_d$ be a basis for S. These should define odd vector fields on the supermanifold $M \times \mathcal{F}$. One can package this data as

$$\delta = \eta^a Q_a$$

where we have adjoined $\eta^1, \ldots, \eta^d$ formal odd parameters. So this even element defines an *even* vector field on $\mathcal{F}$ packaging all of the supersymmetries: one can read off from the action of this element how the supersymmetry corresponding to any individual Q_a acts by looking at the η^a term after applying δ. The use of the symbol δ is intended to be evocative of the first variation from the calculus of variations: it is the tangent vector corresponding to a particular variation of the fields.

This is, in a sense, a *universal* supersymmetry. A general odd element of the supersymmetry algebra looks like $\sum a^i Q_i$, so we must define a supersymmetry for each such element. We should think of defining the above vector field over a universal ring, and specialising to specific values to recover these individual symmetries, such as the action of each individual Q_i.

Now, to check that this defines an action of the super translation algebra, we must first check the commutation relations: i.e. that the action agrees with

$$[\eta_1^a Q_a, \eta_2^b Q_b] = \eta_1^a \eta_2^b \Gamma(Q_a, Q_b).$$

In this universal picture we must clearly work with different universal parameters η_1, η_2, to ensure we check *all* possible supercommutators at once. Secondly, we much check that δ is a symmetry, i.e. that

$$\mathrm{Lie}_\delta \mathcal{L} = d\alpha$$

for some α. Later, we will see this in some examples.

2.3 Superspace picture

A natural setting where we expect a nice action of the super translation algebra T is where spacetime is the corresponding super Minkowski space M, so T acts on spacetime in a natural way. Indeed, many natural field theories in the sense described above are equivalent to such theories on M: in this section we will explain how to recover a supersymmetric field theory on $(\check{M})^n$ from a field theory on M, by taking *component fields.* Throughout this section, $\iota \colon \check{M} \to M$ will denote the inclusion of the even part of superspace, i.e. the map corresponding to the map of super-rings

$$C^\infty(\check{M})[\theta_1, \ldots, \theta_d] \to C^\infty(\check{M})$$

by setting all θ_i to 1.

A standard technique which we will exploit is a choice of basis for the spaces of left and right translation invariant vector fields on M. We fix throughout bases x^a and θ^b for the even and odd parts of the supertranslation algebra T as an $n|d$-dimensional super vector space, acting on M. The vector fields $\partial_i = \frac{\partial}{\partial x^i}$ are both left and right invariant. We can extend this to bases $\partial_1, \ldots, \partial_n, D_1, \ldots, D_d$ and $\partial_1, \ldots, \partial_n, \tau_{Q_1}, \ldots, \tau_{Q_d}$ of left and right invariant vector fields respectively by defining

$$D_a = \partial_a - \theta^b \Gamma^c_{ab} \partial_c$$
$$\tau_{Q_a} = \partial_a + \theta^b \Gamma^c_{ab} \partial_c$$

and checking the appropriate invariances. We use the very different seeming notation for these two families of odd vector fields because we will use them to play different roles. The left invariant vector fields will be used as covariant derivatives, to write down a superspace Lagrangian, while the remaining right invariant vector fields will correspond to the supersymmetries coming from the Q_a. The D_a and τ_{Q_b} commute with one another, and with the even vector fields ∂_i, but there are non-trivial internal supercommutators

$$\frac{1}{2}[D_a, D_b] = -\Gamma^c_{ab}\partial_c$$
$$\frac{1}{2}[\tau_{Q_a}, \tau_{Q_b}] = \Gamma^c_{ab}\partial_c$$

We will call fields on M *superfields*, usually denoted Φ.

Example 2.3. Consider $\mathcal{F} = \mathrm{Map}(M, X)$, for X a Riemannian manifold. This is the space of fields of a *supersymmetric sigma model*. We should take care: this mapping space is itself a supermanifold, so contains more data than just its geometric points. We can analyse the odd data in this space by, for instance, considering the $\mathbb{R}[\eta]$ points of this superspace, for η an odd variable.

Let $\mathcal{L}$ be the Lagrangian in this superspace field theory. Taking component fields roughly corresponds to looking at the Taylor expansion of some Φ in the odd variables. As odd variables are all nilpotent, this Taylor expansion will necessarily terminate after finitely many terms. So, more precisely, fix some $\Phi \colon M \to E$. Then we can expand Φ in local co-ordinates x_i, θ_j as

$$\Phi(x^1, \ldots x^n, \theta^1, \ldots, \theta^d) = \sum_J \iota^* D_J \Phi$$

where the terms use the global frame D_a of left invariant odd vector fields . So D^J represents a product of these D_a's. The right hand side expands as a sum over monomials like $\theta_{a_1} \cdots \theta_{a_k}$, and as the θ_a are odd, there are only finitely many such terms.

In this setting, supersymmetries are easy to see. Indeed, let Q_a be a basis for S, the fermionic part of the super translation algebra. Let η^a be odd parameters. The element $\eta^a Q_a$ of $T[\eta^1, \ldots, \eta^d]$ now generates a diffeomorphism of M using our *right invariant* vector fields τ: namely the exponential $\exp(\eta^a \tau_{Q_a})$. In other words, our vector field acts on superfields by the usual procedure:

$$\delta(\Phi) = \left.\frac{d}{dt}\right|_{t=0} \exp(-t\eta^a \tau_{Q_a})^* \Phi,$$

where the inverse (the minus sign) appears to make this a left action, rather than a right action.

We can produce the corresponding action on each component field $\iota^* D^r \Phi$ explicitly by putting:

$$\delta(\iota^* D^r \Phi) = \left.\frac{d}{dt}\right|_{t=0} \iota^* D^r \left(\exp(-t\eta^a \tau_{Q_a})^* \Phi\right).$$

As long as one is sufficiently careful with what one means by "$\frac{d}{dt}$"after applying the D^r. That is, the supersymmetries are

$$\delta(\iota^* D^r \Phi) = -\eta^a \iota^* D_a D^r \Phi.$$

We have replaced τ_{Q_a} by D_a because we're pulling back to the even part of superspace, on which they agree (see [1] p238), and used the fact that the D_a commutes with D^r.

How can one produce a component Lagrangian from a superspace Lagrangian? The idea is to perform a *Berezin*, or *fermionic integral*, i.e. to push forward under the projection map $\pi \colon M \to \check{M}$. We make a choice of volume form on the fermionic part of superspace, S^*. Call it $d^d\theta$. Then our superspace Lagrangian density has the form

$$\mathcal{L} = |d^n x| d^d\theta \ell,$$

where ℓ is a *function* $\mathcal{F} \to \mathbb{R}$. The density $|d^n x|$ is determined uniquely from the Lorentzian metric on $V = \mathbb{R}^{1,n-1}$ In order to describe a component Lagrangian

$$\mathcal{L}_{\mathrm{com}} = |d^n x| \check{\mathcal{L}}$$

we must integrate out the fermionic directions. In good situations, one can write this component Lagrangian explicitly by choosing a suitable combination of D_a operators, D^r say, and to put $\check{\mathcal{L}} = \iota^* D^r \ell$. Slighly more generally one could add on an appropriate differential operator so that the resulting Lagrangian differs from $\pi_*\mathcal{L}$ by an exact term that will not affect the dynamics. That is:

$$\pi_*\mathcal{L} = \mathcal{L}_{\text{com}} + |d^n x|\Delta\iota^*\ell$$

where Δ is some Poincaré invariant differential operator. We'll see examples of these operators D^r and Δ below: though we'll probably only deal with simple examples where one can choose a D^r so that $\Delta = 0$, i.e. literally replace an integral with a derivative.

Theorem 2.4 (1.36 in [1])**.** This component Lagrangian $\mathcal{L}_{\text{com}}$ is supersymmetric.

Proof. For simplicity, let me only talk about the simple case, where $\mathcal{L}_{\text{com}} = \pi_*\mathcal{L}$. First we show the result of performing Berezin integration – that is, the pushforward $\pi_*\mathcal{L}$ – is supersymmetric. We must check that this Lagrangian is invariant up to an exact term upon Lie differentiation with respect to δ, a vector field on $\check{M} \times \mathcal{F}$ which is *constant* in the $\mathcal{F}$ direction. So we compute, using Cartan's formula,

$$\begin{aligned}
\mathrm{Lie}_\delta\,\mathcal{L} &= \mathrm{Lie}_\delta\,\pi_*\mathcal{L} \\
&= \pi_*\,\mathrm{Lie}_{-\eta^a\tau_{Q_a}}\,\mathcal{L} \\
&= \pi_*\left(d\iota_{-\eta^a\tau_{Q_a}}\mathcal{L}\right) \\
&= d\left(\pi_*\iota_{-\eta^a\tau_{Q_a}}\mathcal{L}\right)
\end{aligned}$$

which is exact as required. □

3 Examples of Supersymmetric Sigma Models

Example 3.1. Let's discuss the *supersymmetric particle*, i.e. the one-dimensional supersymmetric sigma model. We'll give both a superspace description, and a discussion of the component fields.

Let $M = \mathbb{R} \times \Pi S$, where S is the non-trivial irreducible representation of $Spin(1) \cong \mathbb{Z}/2$. So the associated supertranslation algebra is $T = \mathbb{R}^{1|1}$, with bracket $[Q, Q] = 2$, where Q generates the fermionic part. Let (t, θ) be co-ordinates on M. Then we have odd vector fields

$$\begin{aligned}
D &= \partial_\theta - \theta\partial_t \\
\tau_Q &= \partial_\theta + \theta\partial_t.
\end{aligned}$$

Then (∂_t, D) and (∂_t, τ_Q) are bases for the left and right translation invariant vector fields on M respectively, and we have non-trivial commutators

$$\begin{aligned}
[D, D] &= -2\partial_t \\
[\tau_Q, \tau_Q] &= 2\partial_t.
\end{aligned}$$

The *fields* in our sigma model are functions of supermanifolds $\Phi\colon M \to X$, where X is a Riemannian manifold with metric g. By analogy with the classical sigma model, we choose the Lagrangian density

$$\mathcal{L}(\Phi) = -\frac{1}{2}dtd\theta\langle D\phi, \partial_t\Phi\rangle,$$

describing a supersymmetric Lagrangian by our discussion above.

Let's produce the associated component Lagrangian. So we write $\mathcal{L} = dtd\theta\ell$, where ℓ is the Lagrangian function $-1/2\langle D\phi, \partial_t\Phi\rangle$. The component fields are

$$\begin{aligned}
\phi &= \iota^*\Phi \\
\psi &= \iota^* D\Phi,
\end{aligned}$$

where $\iota\colon \mathbb{R} \hookrightarrow M$. So $\phi\colon \mathbb{R} \to X$ is an even function, and ψ is a spinor field in $\Gamma(\mathbb{R}, \phi^*\Pi TX)$. We produce the component Lagrangian by applying the operator $\iota^* D$ to the Lagrangian function above, corresponding to taking the Berezin integral.

$$\begin{aligned}
\mathcal{L}_{\text{com}} &= -\frac{1}{2}dt\ \iota^* D\langle D\Phi, \partial_t \Phi\rangle \\
&= -\frac{1}{2}dt\ \iota^* \left(\langle DD\Phi, \partial_t\Phi\rangle - \langle D\phi, D\partial_t \Phi\rangle\right) \\
&= -\frac{1}{2}dt\ \iota^* \left(\langle -\partial_t\Phi, \partial_t\Phi\rangle - \langle D\phi, \partial_t D\Phi\rangle\right) \\
&= \frac{1}{2}dt \left(|\dot\phi|^2 + \langle \psi, \nabla_{\dot\phi}\psi\rangle\right).
\end{aligned}$$

By the result 2.4, we know this component field Lagrangian is supersymmetric. We can describe the supersymmetries. Indeed, consider the even vector field $\delta = \eta\tau_Q$. Then we compute how δ acts:

$$\begin{aligned}
\delta\phi &= -\eta\iota^* D\Phi = -\eta\psi \\
\delta\psi &= -\eta\iota^* DD\Phi = \eta\iota^*\partial_t\Phi = \eta\dot\phi.
\end{aligned}$$

Example 3.2. For a tougher example, let $M = M^{3|2} = \mathbb{R}^{1,2} \times \Pi S$ where S is the 2d spinor representation of $\mathrm{Spin}(1,2) \cong SL(2,\mathbb{R})$. The super translation algebra T has two odd generators Q_1, Q_2, with non-trivial bracket given by identifying the even piece of the algebra V with $\mathrm{Sym}^2\,\mathbb{R}^2$. So we can choose a basis y_{11}, y_{12}, y_{22} so that the brackets are

$$[Q_1, Q_1] = y_{11},\ [Q_1, Q_2] = y_{12},\ [Q_2, Q_2] = y_{22}.$$

Consider the three dimensional sigma algebra, with fields

$$\mathcal{F} = \{\Phi\colon M^{3|2} \to X\}$$

where again, X is a Riemannian manifold. The Lagrangian density is, similarly to the 1d case, given by

$$\mathcal{L}(\Phi) = \frac{1}{4}|d^3y|d^2\theta\ \varepsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle,$$

where ε comes from the Lie bracket

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and where the left invariant vector fields D_1, D_2 are given by formulae as usual:

$$D_a = \partial_{\theta_a} - \theta^b\partial_{y_{ab}}.$$

We can work out the component fields, and the component Lagrangian as previously. One computes the components

$$\begin{aligned}
\phi &= \iota^*\Phi \\
\psi_a &= \iota^* D_a\Phi \\
F &= -\iota^* DD\Phi
\end{aligned}$$

corresponding to the scalar, θ_a and $\theta_1\theta_2$ terms in the Taylor series for Φ. The field ϕ is a scalar field, i.e. a function $\check{M} \to X$, the ψ_a define an odd spinor field

$$\psi \in \Gamma(\check{M}, \Pi(\phi^*TX \otimes S)),$$

and F is an even section

$$F \in \Gamma(\check{M}, \phi^*TX).$$

On M, we can investigate the operator $D^2 = \frac{1}{2}\varepsilon^{ab}D_aD_b$, which computes the Berezin integral

$$\int d^2\theta = \iota\partial_{\theta_1}\partial_{\theta_2} = \iota^* D^2.$$

This allows us to compute the component Lagrangian. Indeed,

$$\begin{aligned}\mathcal{L}_{\mathrm{com}} &= -\frac{1}{4}\iota^* D^2 \varepsilon^{ab}\langle D_a\Phi, D_b\Phi\rangle \\ &= -\frac{1}{4}\iota^*\varepsilon^{ab}\left(2\langle D_a\Phi, D^2 D_b\Phi\rangle - \varepsilon^{cd}\langle D_c D_a\Phi, D_d D_b\Phi\rangle\right).\end{aligned}$$

Now, we must use simplify $D^2 D_a$ and $D_a D_b$. One checks $D_a D_b = -(\partial_{ab} - \varepsilon_{ab} D^2)$, easily case-by-case, and

$$\begin{aligned}D^2 D_a &= \pm\frac{1}{2} D_1 D_2 D_1 - D_2 D_1 D_1 \\ &= \pm\frac{1}{2} R(D_1, D_2) D_1 - (D_2 D_1) D_1 - 2\nabla_{y_{12}} D_1 - D_2 D_1 D_1 \\ &= \pm\frac{1}{2} R(D_1, D_2) D_1 + 2D_2\partial_{y_{11}} - 2\nabla_{y_{12}} D_1 \\ &= \pm\frac{1}{2} R(D_1, D_2) D_1 + 2\nabla_{y_{11}} D_2 - 2\nabla_{y_{12}} D_1\end{aligned}$$

where the sign depends on the value of a. We used the fact that the curvature R obeys the relation

$$R(D_a, D_b) = D_a D_b + D_b D_a + 2\nabla_{y_{ab}}.$$

We can express this as $D^2 D_a = \varepsilon^{bc}(-\frac{1}{6}R(D_a, D_b) + \nabla_{ab})D_c$. Thus, the component Lagrangian evaluates to

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}\iota^*\varepsilon^{ab}\varepsilon^{cd}\left(\frac{1}{3}\langle D_a\phi, R(D_b\Phi, D_c\Phi)D_d\Phi\rangle - \langle D_a\Phi, \nabla_{bc} D_d\Phi\rangle + \langle\partial_{y_{ca}}\Phi - \varepsilon_{ca}D^2\Phi, \partial_{y_{db}}\Phi - \varepsilon_{db}D^2\Phi\rangle\right) \\ &= \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a\phi^* R(\psi_b, \psi_c)\psi_d\rangle - \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a, \partial_{y_{bc}}\psi_d\rangle + \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\langle\partial_{y_{ca}}\phi + \varepsilon_{ca}F, \partial_{y_{db}}\phi + \varepsilon_{db}F\rangle \\ &= \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a, R(\psi_b, \psi_c)\psi_d\rangle - \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a, \nabla_{y_{bc}}\psi_d\rangle + \frac{1}{2}|d\phi|^2 - \frac{1}{2}|F|^2 \\ &= \frac{1}{2}|d\phi|^2 - \frac{1}{2}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a, \nabla_{y_{bc}}\psi_d\rangle - \frac{1}{2}|F|^2 + \frac{1}{12}\varepsilon^{ab}\varepsilon^{cd}\langle\psi_a, R(\psi_b, \psi_c)\psi_d\rangle\end{aligned}$$

where at the final step we just re-arrange the terms for clarify, so the interaction term is last. The second term here could also be written $\psi \not{D} \psi$, where $\not{D}$ denotes the Dirac form on the relevant spinor representation.

The supersymmetries are easy to compute, namely

$$\begin{aligned}\delta\phi &= -\eta^a\iota^* D\Phi = -\eta^a\psi_a \\ \delta\psi_a &= -\eta^b\iota^* D_b D_a\Phi = \eta^b(\partial_{y_{ab}}\phi - \varepsilon_{ab}F) \\ \delta F &= \eta^a\iota^* D_a D^2\Phi = \eta^a\left(\frac{1}{3}\varepsilon^{bc}R(\psi_a, \psi_b)\psi_c - \varepsilon^{bc}\partial_{y_{ab}}\psi_c\right)\end{aligned}$$

using the expressions for $D_a D_b$ and $D^2 D_a$ derived above.

Note that F will not play any role in the equations of motion (since its derivatives don't appear in the Lagrangian: they will impose that $F = 0$), so it is in some sense a purely auxilliary field.

References

[1] Pierre Deligne and Dan Freed. Supersolutions. In *Quantum Fields and Strings: A Course For Mathematicians*, volume 1. AMS, 1999.