Notes on Supersymmetry Algebras

Chris Elliott

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In this talk I'm going to explain, briefly, what supersymmetry algebras are, and then describe their classification. Finally we'll give a complete description of all possible supersymmetry algebras in low dimensions.

1 Classification of Supersymmetry

The idea of supersymmetry isn't too tricky. Suppose you have a quantum field theory on $\mathbb{R}^n$ depending on a Riemannian metric. We often want to restrict attention to theories where symmetries of $\mathbb{R}^n$ can be lifted to symmetries of the theory. We know exactly what symmetries of $\mathbb{R}^n$ are: the group of isometries.

Definition 1.1. The *Poincaré group* in dimension n is the semi-direct product $\mathrm{ISO}(n) = \mathrm{SO}(n) \ltimes \mathbb{R}^n$. The *Poincaré algebra* is its Lie algebra $\mathfrak{iso}(n) \cong \mathfrak{so}(n) \ltimes \mathbb{R}^n$.

Can we ever have more symmetry than this? A famous theorem of Coleman and Mandula, generalized to higher dimensions by Pelc and Horwitz (more precisely, in dimension $n \geq 4$ and Lorentzian signature) says that, in a nice enough quantum field theory with a Poincaré algebra action, the only way a larger algebra $\mathfrak{iso}(n-1,1) \subseteq \mathfrak{g}$ can act is if it splits as a sum: $\mathfrak{g} \cong \mathfrak{iso}(n-1,1) \oplus \mathfrak{g}'$ (to put this in some context, this is a result about the classification of Hilbert-space representations of Poincaré algebras).

Remark 1.2. In order to avoid issues related to the choice of signature, from now on we'll only consider the *complexified* Lie algebra $\mathfrak{iso}(n;\mathbb{C})$, which is signature independent. There are many interesting things to say about classification of supersymmetry algebras in the presence of a real structure, but we won't get into them today.

We can get around this sort of no-go theorem by considering $\mathbb{Z}/2\mathbb{Z}$-graded Lie algebras (super Lie algebras) which extend $\mathfrak{iso}(n;\mathbb{C})$ in a non-trivial way.

Definition 1.3. An n-dimensional *super Poincaré algebra* is a super Lie algebra

$$\mathfrak{A} = \mathfrak{iso}(n;\mathbb{C}) \oplus \Pi\Sigma$$

whose even part is the complexified Poincaré algebra, and whose odd part is a spinorial representation of $\mathfrak{so}(n;\mathbb{C})$, i.e. a representation none of whose irreducible summands lift to representations of the group $\mathrm{SO}(n;\mathbb{C})$ [1].

Luckily, this definition is quite restrictive: we can list all the possible super Poincaré algebras in every dimension. First, let's classify irreducible spinorial representations of $\mathfrak{so}(n;\mathbb{C})$. We'll describe the classification without proof.

Lemma 1.4. If n is odd, there is a unique spin representation S (the Dirac spinor representation) of dimension $2^{\frac{n+1}{2}}$. If n is even, there are two non-isomorphic spin representations S_+ and S_- (Weyl spinor representations – the Dirac spinor is $S_+ \oplus S_-$) both of dimension $2^{\frac{n}{2}}$.

[1] This definition isn't quite accurate when $n < 3$, there we really need to classify the irreducible spinorial representations explicitly: all representations when $n = 1$ and all representations with odd charge for $n = 2$.

Remark 1.5. These representations can be described explicitly using Clifford algebras $C^+(n)$ into which the spin groups embed. One can abstractly identify the even part $C^+(n) \subseteq C(n)$ of the n-dimensional complex Clifford algebra with either a matrix algebra (in odd dimensions) or a sum of two matrix algebras (in even dimensions). The spin representations come from the defining representations of these matrix algebras. With a bit of work one can characterize these isomorphisms pretty explicitly.

As a result, we can describe the odd part of the supersymmetry algebra as follows.

- When n is odd, a general spinorial representation takes the form $W \otimes S$ for an $\mathcal{N}$-dimensional complex vector space W.
- When n is even, a general spinorial representation takes the form $W_+ \otimes S_+ \oplus W_- \otimes S_-$ for a pair of complex vector spaces (W_+, W_-) of dimensions $(\mathcal{N}_+, \mathcal{N}_-)$.

The remaining step is to classify the possible brackets between two odd vectors. That is, in the odd and even cases respectively, we need to classify $\mathfrak{so}(n;\mathbb{C})$-equivariant symmetric maps

$$\Gamma \colon (S \otimes W) \otimes (S \otimes W) \to \mathbb{C}^n$$
$$\text{and } \Gamma \colon (S_+ \otimes W_+ \oplus S_- \otimes W_-) \otimes (S_+ \otimes W_+ \oplus S_- \otimes W_-) \to \mathbb{C}^n.$$

Remark 1.6. The structure constants of the pairing Γ can be identified with the physicists' γ-*matrices*, i.e.

$$\Gamma(Q, Q')^i = \gamma^i_{ab} Q^a Q^b.$$

The γ-matrices satisfy Clifford relations of the form $\gamma^i\gamma^j + \gamma^j\gamma^i = 0$ for $i \neq j$. In fact they are equivalent to the structure constants for the Clifford multiplication using some duality moves (and canonical bilinear pairings) to identify maps of the form $S \otimes S \to \mathbb{C}^n$ with $\mathbb{C}^n \otimes S \to S$.

We can perform this classification using the following facts (a version of Bott periodicity), noting that an equivariant morphism to V is the same as the identification of a summand isomorphic to V. As a result, to classify pairings it's enough to decompose the tensor square of S or $S_+ \oplus S_-$ into irreducible summands. Again, I won't prove it, but we'll work through some examples.

Lemma 1.7. When n is odd there is a unique irreducible summand of $S^{\otimes 2}$ isomorphic to $\mathbb{C}^n$. It is contained in $\mathrm{Sym}^2(S)$ if $n \equiv 1, 3 \mod 8$ and in $\wedge^2(S)$ otherwise.

When n is divisible by 4 there is a unique irreducible summand of $S_+ \otimes S_-$ isomorphic to $\mathbb{C}^n$, and no such summand in $S_\pm^{\otimes 2}$.

When $n \equiv 2 \mod 4$ there is a unique irreducible summand of $S_+ \otimes S_+$ and of $S_- \otimes S_-$ isomorphic to $\mathbb{C}^n$, and no such summand in $S_+ \otimes S_-$. It is contained in $\mathrm{Sym}^2 S_\pm$ if $n \cong 2 \mod 8$ and in $\wedge^2 S_\pm$ if $n \cong 6 \mod 8$.

Remark 1.8. We can say something more precise (and this is how you prove the above Lemma). If n is odd then we can identify (using that S is self-dual)

$$\begin{aligned} S \otimes S \cong C^+(V) &\cong \bigoplus_{k \text{ even}} \wedge^k(V) \\ &\cong \bigoplus_{k=0}^{\frac{n-1}{2}} \wedge^k(V). \end{aligned}$$

Likewise, if n is even and $S = S_+ \oplus S_-$ then we can decompose (using that S_+ and S_- are mutually dual)

$$\begin{aligned} S \otimes S \cong C(V) &\cong \bigoplus_{k=0}^{n} \wedge^k(V) \\ &\cong 2\left(\bigoplus_{k=0}^{n/2-1} \wedge^k(V)\right) \oplus \wedge^{n/2}(V). \end{aligned}$$

One then has to work a little more to understand the splitting into tensor products of S_+ and S_-.

The upshot of this story is that the choice of a super Poincaré algebra is a choice of

- A single orthogonal vector space W if $n \equiv 1, 3 \mod 8$.
- A pair of orthogonal vector spaces W_+, W_- if $n \equiv 2 \mod 8$.
- A single vector space $W_+ = W$, with dual $W_- = W^*$, if $n \equiv 0, 8 \mod 8$.
- A single symplectic vector space W if $n \equiv 5, 7 \mod 8$.
- A pair of symplectic vector spaces W_+, W_- if $n \equiv 6 \mod 8$.

Remark 1.9 (On Terminology)**.** So now we've abstractly classified the possible super Poincaré algebras, we should give them names. It's traditional to indicate a particular supersymmetry algebra using the dimension $\mathcal{N}$ or dimensions $(\mathcal{N}_+, \mathcal{N}_-)$ of the auxiliary space(s) W or (W_+, W_-). So you refer to things like "the 3-dimensional $\mathcal{N} = 2$ supersymmetry algebra".

There are exceptions however. When $W \equiv 5, 6, 7 \mod 8$, so the auxiliary spaces are symplectic, we instead use $\mathcal{N}$ to indicate *half* the dimension of the auxiliary space, so that $\mathcal{N} = 1$ is the minimal supersymmetry algebra. This is a little confusing, but it's the standard convention.

Finally, one sometimes indicates supersymmetry algebras using the "number of supersymmetries", meaning the dimension of the odd part of the supersymmetry algebra. We often restrict attention to either algebras with at most 16 supercharges, or at most 32 supercharges. The reason for this is representation-theoretic: in dimensions at least 4 these are the only super Poincaré algebras admitting non-trivial (massive) representations with spins at most 1 (16 supercharges) or at most 2 (32 supercharges). The former restriction arises in supersymmetric gauge theory, and means the dimension $n \leq 10$, and the latter in supergravity, and means the dimension $n \leq 11$.

Definition 1.10. The *R-symmetry group* G_R of a supersymmetry algebra $\mathfrak{A}$ is the group of outer automorphisms of $\mathfrak{A}$ that act trivially on the even part. They are just given by automorphisms of the auxiliary vector spaces W, W_+, W_- that preserve the given structure (so usually they are products of orthogonal or symplectic groups, except when $n \equiv 0, 4 \mod 8$ when we get the R-symmetry group $\mathrm{GL}(W)$).

1.1 Square Zero Supercharges

I just want to take a moment to mention square zero supercharges and their classification.

Definition 1.11. We say a supercharge Q in the odd part of a super Poincaré algebra is *square zero* if $\Gamma(Q, Q) = 0$.

These symmetries are important because they define cohomological structures (i.e. cochain complex structures) on spaces like the algebra of observables, or the Hilbert space, of a classical and quantum field theory. If the map $\Gamma(Q, -)$ is surjective onto $\mathbb{C}^n$ is surjective then we say Q is *topological.* The cohomological structures associated to topological square-zero supercharges allow us to define the notion of a *topological twist* of a supersymmetric quantum field theory – typically such theories will be topological.

Remark 1.12. If a subgroup $G \subseteq \mathrm{Spin}(n) \times G_R$ leaves a square-zero supercharge invariant, then the action of the group G will survive to the twist. For instance, if we want $\mathrm{Spin}(n)$ to continue to act after twisting, we might try embedding $\mathrm{Spin}(n)$ diagonally in the product $\mathrm{Spin}(n) \times G_R$ using a homomorphism $\phi \colon \mathrm{Spin}(n) \to G_R$. Such homomorphisms show up all the time when one studies twists of supersymmetric field theories. Such a ϕ can be found for topological twists in dimensions less than 6.

2 Examples

1. **Dimension 1:** Since $\mathfrak{so}(1; \mathbb{C})$ is trivial, the supersymmetry algebra is just $\mathbb{C} \oplus \Pi W$ where W is an orthogonal vector space of dimension $\mathcal{N}$. So, the R-symmetry group will be $\mathrm{O}(\mathcal{N})$. Square-zero supercharges are equivalent to null vectors in W, i.e. elements w such that $(w, w) = 0$.

For example, if $\mathcal{N} = 2$, there are null vectors of the form $(1, i)$ and $(1, -i)$. The conserved currents of these supersymmetries are the Q and $Q^\dagger$ we saw in Matthew's talk yesterday.

2. **Dimension 2:** Now $\mathfrak{so}(2;\mathbb{C}) \cong \mathbb{C}^\times$. The vector representation $\mathbb{C}^2$ has weights 1 and -1. The two Weyl spinor representations $S_\pm$ are 1-dimensional, with weights $1/2$ and $-1/2$ respectively, with the obvious pairing to the two irreducible summands of the representation $\mathbb{C}^2$. There is a supersymmetry algebra associated to a pair of orthogonal vector spaces $W_\pm$ of dimensions $\mathcal{N}_\pm$. It has R-symmetry group $\mathrm{O}(\mathcal{N}_+) \times \mathrm{O}(\mathcal{N}_-)$.

   Square-zero supercharges are equivalent to pairs of null vectors $(w_+, w_-) \in W_+ \oplus W_-$. They are topological if w_+, w_- are both non-zero, and holomorphic otherwise.

3. **Dimension 3:** In this example $\mathfrak{so}(3;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C})$: the spinor representation S is the 2-dimensional defining representation and the vector representation is isomorphic to $\mathrm{Sym}^2(S)$, with the obvious pairing. We have a supersymmetry algebra associated to an orthogonal vector space W of dimension $\mathcal{N}$, again with R-symmetry group $\mathrm{O}(\mathcal{N})$.

   A supercharge $Q \in S \otimes W$ defines a linear map $S^* \to W$. It squares to zero exactly when the image is totally isotropic (i.e. the restriction of the metric to the image is zero. You can only get a topological supercharge when $\mathcal{N} \geq 4$.

4. **Dimension 4:** In this example $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_-$. The two Weyl spinor representations $S_\pm$ are identified with the two defining representations of the factors, and the vector representation is isomorphic to their tensor product $S_+ \otimes S_-$. So here, the supersymmetry algebra looks like

$$\mathfrak{sl}(2;\mathbb{C})_+ \oplus \mathfrak{sl}(2;\mathbb{C})_- \oplus \mathbb{C}^4 \oplus \Pi(S_+ \otimes W \oplus S_- \otimes W^*),$$

   where W has dimension $\mathcal{N}$, and the R-symmetry group is $\mathrm{GL}(\mathcal{N})$.

   Here square-zero supercharges are the same as pairs of subspaces W_{Q_+}, W_{Q_-} of W and W^* respectively of dimension ≤ 2, which pair to zero. They are topological when at least one of the subspaces has dimension 2.

5. **Dimension $\geq$ 5:** (For reference) In dimension 5 the spinor splits as $\wedge^2 S \cong \mathbb{C}^5 \oplus \mathbb{C}$ and square zero spinors are a bit of a pain to classify concisely (under $Q\colon W^* \to S$ the Poisson bivector pulls back to the Poisson bivector). In dimension 6 $\wedge^2(S_\pm) \cong \mathbb{C}^6$ exactly. Square zero is again a bit of a pain: the two spaces $W_{Q_\pm}$ define an extension isomorphic to S_+, with all maps preserving volume forms. Beyond this you can calculate, but the square zero conditions get more complicated (though they aren't bad for $\mathcal{N} = 1$).