What is Supersymmetry?

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Introduction

I'm going to tell a story about algebra whose motivation comes from quantum physics. Particles are classified into two types: bosons and fermions. These particles behave somewhat differently: in quantum mechanics a state of a system is described by a wavefunction ψ.

- If you interchange two bosons, the wavefunction stays the same $\psi \mapsto \psi$.
- If you interchange two fermions, the wavefunction picks up a sign $\psi \mapsto -\psi$.

This is a talk about "supermathematics", where we study algebraic objects that split up into bosonic and fermionic pieces. Supermathematics is used to describe physical systems containing both fermions and bosons, and supersymmetries are symmetries that swap bosons and fermions.

Superalgebra

A super vector space is just a vector space that splits up into even and odd pieces:

$$V = V_0 \oplus V_1.$$

We talk about V having a $\mathbb{Z}/2\mathbb{Z}$-grading, and we say an element $v \in V_j$ has degree j (written $|v| = j$).

A commutative super algebra is a super vector space A with an associative product, where

$$|a \cdot b| = |a| + |b|,$$

and

$$a \cdot b = (-1)^{|a||b|} b \cdot a.$$

Example

Let's talk about free superalgebras over $\mathbb{C}$.

- $\mathbb{C}[\varepsilon]$ where ε is odd. Here $\varepsilon^2 = 0$, so $\mathbb{C}[\varepsilon] = \mathbb{C} \oplus \varepsilon\mathbb{C}$ is 1 $1 + 1$-dimensional super vector space.
- $\mathbb{C}[\varepsilon_1, \cdots, \varepsilon_n]$ where the ε_i are all odd. All the ε_i square to zero, so this algebra has dimension 2^n as a vector space. Can identify with the cohomology of $(S^1)^n$.
- $\mathbb{C}[x_1, \cdots, x_n, \varepsilon_1, \cdots, \varepsilon_n]$ where the x_i are even and the ε_i are odd. The x's commute with the ε's: this is a module over the ordinary polynomial ring of rank 2^n. This gives a model for polynomial differential forms on $\mathbb{C}^n$.

Lie Algebras

Our next example of super algebra will generalize the following idea from non-super mathematics.

Definition

A Lie algebra is a vector space L equipped with a bilinear product $[,]$ that we call a Lie bracket that is:

- Antisymmetric: $[a, b] = -[b, a]$.
- Jacobi: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

Lie algebras can be used to describe "infinitesimal symmetry" in geometry. If you have a Lie group G acting on a manifold M, you can "differentiate" the action to find a Lie algebra acting on the tangent spaces of M.

Examples

Let's give a few examples, to be generalized to the super case shortly.

- Abelian Lie algebras: $\mathbb{C}^n$ with the zero bracket.
- Define $\mathfrak{gl}(n;\mathbb{C})$ to be the vector space of $n \times n$ complex matrices, with bracket given by commutator. This Lie algebra describes all linear (infinitesimal) automorphisms of $\mathbb{C}^n$.
- Define $\mathfrak{so}(n;\mathbb{C}) \subseteq \mathfrak{gl}(n;\mathbb{C})$ to be the subalgebra consisting of those matrices A where $A^T = -A$ (exercise: check this is a vector subspace closed under the bracket). This Lie algebra describes infinitesimal rotations of an n-dimensional vector space.

Super Lie Algebras

Definition

A super Lie algebra is a supervector space L equipped with a product $[,]$ such that $|[a,b]| = |a| + |b|$, that is

- Super antisymmetric: $[a,b] = (-1)^{|a||b|+1}[b,a]$.
- Super Jacobi:

$$(-1)^{|a||c|}[a,[b,c]]+(-1)^{|a||b|}[b,[c,a]]+(-1)^{|b||c|}[c,[a,b]] = 0.$$

Examples

- Write $\mathbb{C}^{n|m}$ for the super vector space with even part $\mathbb{C}^n$ and odd part $\mathbb{C}^m$. We define $\mathfrak{gl}(n|m;\mathbb{C})$ to be the super vector space of all linear maps $\mathbb{C}^{n|m} \to \mathbb{C}^{n|m}$. We give it the "supercommutator" Lie bracket, defined on homogenous elements by

$$[A, B] = AB + (-1)^{|A||B|+1} BA.$$

- There's a super analogue of $\mathfrak{so}(n)$ denoted $\mathfrak{osp}(n|m)$ (the orthosymplectic super Lie algebra).

Super Translation

Finally, let's talk about our main kind of example. We'll consider super translation algebras. That is, super Lie algebras T whose even part is just an abelian Lie algebra $\mathbb{C}^n$. So as a super vector space,

$$T = \mathbb{C}^n \oplus \Pi S$$

for some S (here Π means parity shift, i.e. placing in odd degree). The only bracket is a bilinear symmetric map

$$\Gamma \colon S \otimes S \to \mathbb{C}^n.$$

We'd like the Lie algebra $\mathfrak{so}(n;\mathbb{C})$ to act on T, extending the action on $\mathbb{C}^n$ by rotations. So that means we have to specify an action on S, and we need Γ to be equivariant (to intertwine the actions on the source and target).

Examples

- Dimension 3: $\mathfrak{so}(3;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C})$. Let $S = V \otimes W_{\mathcal{N}}$ where V is the 2d defining representation and $W_{\mathcal{N}}$ is $\mathcal{N}$-dimensional, with trivial action. There is a unique pairing $\Gamma \colon V \otimes V \cong \mathbb{C}^3$.
- Dimension 4: $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{sl}(2;\mathbb{C}) \oplus \mathfrak{sl}(2;\mathbb{C})$. Let $S = V_+ \otimes W_{\mathcal{N}} \oplus V_- \otimes W_{\mathcal{N}}^*$ where $V_\pm$ are the 2d defining representation of the two factors and $W_{\mathcal{N}}$ is $\mathcal{N}$-dimensional, with trivial action. There is a unique pairing $\Gamma \colon V_+ \otimes V_- \cong \mathbb{C}^4$.

Application to Physics

One starting point for models of classical and quantum field theory involves the following data:

- A super vector space Φ of fields (usually the space of sections of a vector bundle on $\mathbb{R}^n$).
- A functional $S\colon \Phi \to \mathbb{C}$ called the action; physical states are critical points of this action.

A symmetry of the system is a linear map $\Phi \to \Phi$ that leaves S invariant. Under some assumptions on the physical system, the possible symmetries are quite restricted.

Theorem (Coleman–Mandula)

For a "nice enough" quantum field theory, the group of symmetries is a direct product $\mathrm{ISO}(n) \times G$.

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One way around this theorem is to generalize the notion of symmetry from ordinary groups to supergroups. That is, to include supersymmetries that interchange bosons and fermions. There are "nice" field theories that carry an action of supertranslation algebras. Such field theories are called supersymmetric.

Thank you for listening!