Donaldson and Seiberg-Witten theory and their relation to $\mathcal{N}=2$ SYM

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April 23, 2013

We've began to see what it means to twist a supersymmetric field theory. I will review Donaldson theory and Seiberg-Witten theory, which have applications to the classification of four-manifolds. Then I will hint at a relation of these classification theories to a twist of a particular $\mathcal{N}=2$ $4d$ supersymmetric Yang-Mills theory. Hopefully Kevin or someone else will go further into this.

A quick review of Donaldson theory

A major problem in topology and differential geometry is the classification of smooth manifolds. Donaldson came up with a program that assigns *smooth* invariants to manifolds, and shed some light into the craziness of dimension four. There is a lot of hard analysis going on in the background here, but I will choose to avoid that for sake of exposition.

Here is the setup. Let M be a closed, simply connected, oriented, Riemannian manifold. Let $P \to M$ be a principal SU(2) bundle with connection ω. Let V be the irreducible two-dimensional representation of SU(2). We get a 2-plane bundle $E = P \times_{\mathrm{SU}(2)} V$ over M, and an induced connection ∇ on E. Locally, $\nabla = d + A$, and the curvature has the form $F = F_\nabla = dA + \frac{1}{2} A \wedge A$. Now $F \wedge F$ defines an element of $\Omega^4(M; \mathfrak{su}(2))$ and we may take the trace of this. It is a basic fact from Chern-Weil theory that

$$\int_M c_2(E)\, \mathrm{dvol}_g = -\frac{1}{8\pi} \int_M \mathrm{tr}(F \wedge F)\, \mathrm{dvol}_g.$$

Consider the nasty group $\mathcal{G} = \mathrm{Aut}(P)$ of automorphisms of P. Let $\mathcal{A}'$ be the (infinite dimensional) vector space of connections on P. Then $\mathcal{G}$ acts on $\mathcal{A}'$ in the obvious way. This action has huge stabilizers and one could never hope the quotient would look nice. By choosing appropriate Sobolev spaces one can make the quotient into a Hilber manifold. Donaldson theory is concerned with a subquotient of this space.

Now, recall the Hodge star operator in this context:

$$\star : \Omega^i(M; E) \to \Omega^{4-i}(M; E).$$

Since $\star^2 = \mathrm{id}$ on 2-forms we have a splitting

$$\Omega^2(M; V) = \Omega^2_+(M; V) \oplus \Omega^2_-(M; V)$$

called the self-dual (eigenvalue $+1$) and anti-self-dual (eigenvalue -1) 2-forms. Donaldson theory is concerned with anti-self-dual forms, as these are the natural objects that arise in the holomorphic setting, even thought there is no a priori reason to consider them over the self-dual forms in the real case. Clearly the subspace $\mathcal{A} \subset \mathcal{A}'$ of ASD connections is invariant under the action of $\mathcal{G}$ and we may hence define (using, of course, appropriate Sobolev spaces)

$$\mathcal{M} := \mathcal{A}/\mathcal{G}$$

the moduli space of ASD connections. A lot goes into checking the moduli space is locally well behaved, but one can show for generic metrics $\mathcal{M}$ (or a slight variant of) is a smooth manifold of dimension $d = 8c_2(E) - 3(b_0 - b_1 + b_2^+)$, atleast when $b_2^+ > 0$. Moreover, one can show that for a generic path of metrics the moduli space $\mathcal{M}(P, g_t) = \{([A], t)\}$ is a smooth oriented manifold.

Rougly, Donaldson invariants are defined as integrals of differential forms on the moduli space of irreducible ASD connections. As in the case with invariant theory, we need to define some universal bundle. Let $\mathcal{Q}$ be the principal bundle $\mathcal{A}(P) \times_{\mathcal{G}} P$ over $\mathcal{B}(P) \times M$. Then $p_1(\mathcal{Q}) \in H^4(\mathcal{B}(P) \times M)$ and we may take the slant product

$$H_2(M) \to H^2(\mathcal{B}(P)) \ , \ x \mapsto p_1(\mathcal{Q})/x.$$

It is uniquely determined by the homology-cohomology pairing relation

$$\langle \alpha/x, y \rangle = \langle \alpha, x \times y \rangle$$

where $x \times y$ is the homology cross product. It is a fact that $p_1(\mathcal{Q})/x$ is divisble by 4 so we may define

$$\mu(x) := -\frac{p_1(x)/x}{4} \in H^2(\mathcal{B}).$$

We can extend this to a map

$$\overline{\mu} : H_2(M) \to H^2(\mathcal{M}(P)^+)$$

which is very nontrivial. When $d \equiv \frac{3}{2}(1 + b_2^+) \mod 4$ we can define

$$\gamma_d : H^2(B)^{\otimes d} \to \mathbb{Z} \quad , \quad x_1 \otimes \cdots \otimes x_d \mapsto \int_{\mathcal{M}(P)} \overline{\mu}(x_1) \wedge \cdots \wedge \overline{\mu}(x_d).$$

Seiberg-Witten Theory

Donaldson theory provided a beautiful program for assigning smooth invariants to manifolds and allowed for many classification theorems. There are some drawbacks, however. The moduli space is not as well behaved as we would like it. For instance, it is not compact. This stems from the complexity of the equations of motion defining the moduli space. In 1994, Witten provided a much simpler set of equations that produce a moduli space and invariants that hold essentially the same information. In this section I will give the mathematical program for producing such equations and invariants. In the last section we will briefly look at the field theory that inspired all of this.

Let's set up SW theory. Want to fix a smooth, oriented, four-manifold M with a so-called spin^c structure.

What does this mean? If $\widetilde{G} \to G$ is any covering space of Lie groups and $P \to M$ is a principal G-bundle, we can talk of a lift of P to a principal $\widetilde{G}$ bundle as follows. It is a principal $\widetilde{G}$ bundle $\widetilde{P} \to M$ with a bundle map $\widetilde{P} \to P$ that is equivariant with respect to the covering map $\widetilde{G} \to G$. In our case, we consider the double cover $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. A spin structure on M is a lift of the principal $\mathrm{SO}(n)$ bundle of orthonormal frames to a $\mathrm{Spin}(n)$ bundle $\widetilde{P} \to M$. A basic argument with the Serre spectral sequence shows that

$$M \text{ is spin} \Leftrightarrow w_2(TM) = 0$$

and in this case, spin structure are in bijective correspondence with $H^1(M; \mathbb{F}_2)$.

Example 2.1. For any field k, the total Steifel-Whitney class of $k\mathbb{P}^n = (1+a)^{n+1}$ where a is a generator of the first non-vanishing cohomology group. For $k = \mathbb{R}$, we see that $\mathbb{R}P^n$ is spin iff $n \equiv 3 \mod 4$. For $k = \mathbb{C}$ we see that $\mathbb{C}P^n$ is spin iff n is odd. This shows, in particular, that $\mathbb{C}P^2$ is an oriented four-manifold that is not spin.

The objects we would like to consider depend on a spin structure being present, which is an issue as not all four-manifolds admit such a structure. It is useful to consider a slightly weaker structure; of a manifold being spin^c. Recall,

$$\mathrm{Spin}^c(n) := \mathrm{Spin}(n) \times_{\mathbb{Z}/2} \mathrm{U}(1).$$

Now, we may talk of lifting the bundle of orthornormal frames to a spin^c structure in multiple, albeit equivalent ways. From the definition, we get a short exact sequence

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \times \mathrm{U}(1) \longrightarrow 1. \tag{1}$$

A *spin^c* structure on M is a U(1)-bundle $Q \to M$ and a lift of $P_{TM} \times Q$ to a principal $\mathrm{Spin}^c(n)$ bundle $\widetilde{P}^c \to M$. That is, there is a bundle map $\widetilde{P}^c \to P_{TM} \times Q$ that is equivariant wrt the rightmost map in (1). Alternatively, we have a short exact sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1. \tag{2}$$

We may also define a spin^c structure as a lift of P_{TM} to a principal $\mathrm{Spin}^c(n)$ bundle. That is, there is a bundle map $\widetilde{P} \to P$ that is equivariant wrt the rightmost map in (2). It turns out these two definitions are equivalent, and the resulting spin^c structures are related in an obvious way. Namely, if we are given $\widetilde{P}$ we can form a line bundle $L_\sigma := \widetilde{P} \times_\lambda \mathbb{C}$ where $\lambda(z) = z^2$. The $U(1)$-frames of this bundle exactly corresponds to Q.

It is a basic fact that spin^c structures exist on M iff $w_2(TM)$ is the reduction of a integral homology class. It is a basic fact that this is always the case for four-dimensional manifolds. That is, four-dimensional manifolds always admit spin structures. Thus $\mathbb{C}P^2$, for instance, is a manifold that is spin^c but not spin.

Example 2.2. (Spin $\Rightarrow$ Spin^c) Suppose we are given a spin structure $\widetilde{P}_0 \to M$. Let Q_0 be the trivial U(1) bundle over M. Form

$$\widetilde{P} = \widetilde{P}_0 \times_{\mathbb{Z}/2} Q_0$$

where $\mathbb{Z}/2$ acts as $(p,z) \sim (-p,-z)$. Then $\widetilde{P}$ is a $\mathrm{Spin}^c(n)$ bundle by the action $[(g,\lambda)] \cdot [(p,z)] := [(gp, \lambda z)]$, and clearly is equivariant with respect to the covering map.

Example 2.3. Every complex manifold admits a canonical spin^c structure. Maybe later.

Now, recall the complex spin representation. In dimension 4 it splits into two irreducible reps:

$$\mathrm{Spin}(4) \to \mathrm{GL}(S^\pm)$$

where $\dim S^\pm = 2$. It is a fact that this spin rep uniquely extends to a rep of spin^c

$$\Delta : \mathrm{Spin}^c(4) \to \mathrm{GL}(S^\pm).$$

Denote by $S = S^+ \oplus S^-$. We may also interpret the spin group as sitting inside the the even part of the Clifford algebra $\mathrm{C}\ell(n)$. Likewise, $\mathrm{Spin}^c(n)$ embeds in the even part of the complexified clifford algebra. The spin representation above is just a restriction of Clifford multiplication.

Finally, let M be an oriented four-manifold with spin^c structure $\sigma = (\widetilde{P}, Q)$. Define the *spinor bundles* using the above rep:

$$S^{\pm}_{\sigma} := \widetilde{P} \times_{\Delta} S^{\pm}.$$

Let $S_\sigma = S^+_\sigma \oplus S^-_\sigma$. We also have Clifford algebra bundles

$$\mathrm{C}\ell^{\mathbb{C}}_{\sigma} := P \times_{\mathrm{SO}(4)} (\mathrm{C}\ell(4) \otimes \mathbb{C}).$$

Clifford multiplication $(\mathrm{C}\ell(4) \otimes \mathbb{C}) \otimes S \to S$ extends to a bundle map

$$\mathrm{cliff} : \mathrm{C}\ell^{\mathbb{C}}_{\sigma} \otimes S \to S$$

and the odd part of $\mathrm{C}\ell^{\mathbb{C}}_{\sigma}$ maps $S^{\pm}$ to $S^{\mp}$, as it should.

Fix the Levi-Civita connection on P_{TM} and let A be a connection on Q. This data gives a connection on $\widetilde{P}$ and we can associate connections $\nabla^{\pm}_A$ to $S^{\pm}_{\sigma}$ respectively. Denote by $\not{\partial}^{\pm}_A$ the compositions:

$$\Gamma(S^{\pm}) \xrightarrow{\nabla^{\pm}_A} \Gamma(S^{\pm} \otimes T^*M) \xrightarrow{\mathrm{cliff}} \Gamma(S^{\mp}).$$

These are the plus and minus Dirac operators. The odd sum of these

$$\not{\partial}_A = \begin{pmatrix} 0 & \not{\partial}^-_A \\ \not{\partial}^+_A & 0 \end{pmatrix}.$$

In local coordinates the Dirac operator is not scary. Fix a trivialization of S_σ about a point and let e_i be a basis. Then

$$\begin{aligned} \partial_A \psi|_x &= \sum_i e_i \cdot \nabla_{e_i}(\psi)|_x \\ &= d\psi|_x + \sum_i \left(A(e_i)e_i + \sum_{j<k} \omega_{k,j}(e_i)(e_j e_k) \right) \cdot \psi(x). \end{aligned}$$

Of course, it can be viewed as a twisted exterior derivative.

Now, we are ready to talk about the SW equations and SW moduli space. Our objects will be

$$\mathcal{C}_\sigma = \mathcal{A}_\sigma \times \Gamma(S^+_\sigma)$$

where $\mathcal{A}_\sigma$ is the affine space of unitary connections on Q. Fixing A_0, we can write $\mathcal{A}_\sigma = A_0 + \Omega^1(M) \otimes i\mathbb{R}$. Make the canonical identification of $\bigwedge^2_+ T^*M \otimes \mathbb{C}$ with the traceless endomorphisms of S^+_σ. For $\psi \in \Gamma(S^+_\sigma)$ consider the assigment

$$\Gamma(S^+_\sigma)\varphi \mapsto \langle \psi | \varphi \rangle \, \psi - \frac{1}{2} \langle \psi | \psi \rangle \, \varphi.$$

One checks that this defines a traceless endomorphism $q(\psi)$ of S^+_σ, hence determines an element of $\Omega^2_+(M; i\mathbb{R})$. Choosing a basis and writing $\psi = (\psi_1 \;\; \psi_2)$ the endomorphism $q(\psi)$ has matrix representation:

$$\begin{pmatrix} \frac{1}{2}(|\psi_1|^2 - |\psi_2|^2) & \psi_1 \overline{\psi}_2 \\ \overline{\psi}_1 \psi_2 & -\frac{1}{2}(|\psi_1|^2 - |\psi_2|^2). \end{pmatrix}$$

From which it is immediate that it is traceless. We define the *Seiberg-Witten equations* for (A,ψ) to be

$$F_A^+ = q(\psi) \;\; , \;\; \partial\!\!\!/_A^+ \psi = 0.$$

Now I need to tell you what the gauge group is. We choose

$$\mathcal{G} = \mathrm{Aut}_0(\widetilde{P})$$

the group of automorphisms of $\widetilde{P}$ covering the identity on M. There is a canonical identification of $\mathcal{G}$ with $\mathrm{Maps}(M,S^1)$ which goes as follows. For any principal G-bundle $P \to M$ we have the identification

$$\mathrm{Aut}(P) \longleftrightarrow \{\widehat{\gamma} : P \to G \mid \widehat{\gamma}(pg) = \mathrm{Ad}_{g^{-1}}(\widehat{\gamma}(p)) \equiv g^{-1}\widehat{\gamma}(p)g\}$$

induced from the relation $\gamma(p) = p\widehat{\gamma}(p)$. For our case, let's write $\gamma(\widetilde{p}) = \widetilde{p}\widehat{\gamma}(\widetilde{p})$. We have

$$\pi_1(\widetilde{p}\widehat{\gamma}(\widetilde{p})) = \pi_1(\widetilde{p})\rho^c(\widehat{\gamma}(\widetilde{p}))$$

So, for $\gamma \in \mathcal{G}$, we must have $\widehat{\gamma}(\widetilde{p}) \in \ker \rho^c = \mathrm{U}(1) \subset \mathrm{Spin}^c(4)$ for all $\widetilde{p} \in \widetilde{P}$.

Since $\mathrm{U}(1)$ is the center of $\mathrm{Spin}^c(4)$, if $\widehat{\gamma}(\widetilde{p}) \in U(1)$ then $g^{-1}\widehat{\gamma}(\widetilde{p})g = \widehat{\gamma}(\widetilde{p})$, so $\widehat{\gamma}$ is constant on fibers of $\widetilde{P} \to M$. Hence $\widehat{\gamma}$ descends to a map $f_\gamma : M \to \mathrm{U}(1)$. It is easy to see this argument works backwards, so we have established:

$$\mathcal{G} \longleftrightarrow \mathrm{Maps}(M,S^1)$$

We denote this correspondence on elements by $\gamma \longleftrightarrow f_\gamma$.

The action is defined as:

$$\mathcal{C} \times \mathcal{G} \to \mathcal{C} \;,\; (A,\psi)\cdot\gamma := ((\det\gamma)^* A, (S^+\gamma^{-1})\cdot\gamma).$$

Lemma 2.1. *Let $(A,\psi) \in \mathcal{C}$ and $\gamma \in \mathcal{G}$. Then*

$$\partial\!\!\!/_{(\det\gamma)^*A}(S^+(\gamma^{-1})\cdot\psi) = S^-(\gamma^{-1})\cdot(\partial\!\!\!/_A\psi).$$

Proof. Note this is equivalent to proving $\partial\!\!\!/_{A+2f_\gamma^{-1}df_\gamma}(f_\gamma^{-1}\psi) = f_\gamma^{-1}\partial\!\!\!/_A\psi$. At a point x, computing in terms of an orthonormal basis $\{e_i\}$ of T_xM determined by a section $\widetilde{s}$ on a small neighborhood U of x, we have

$$\begin{aligned}
\partial\!\!\!/_{A+2f_\gamma^{-1}df_\gamma}(f_\gamma^{-1}\psi) - f_\gamma^{-1}\partial\!\!\!/_A\psi &= \Bigg\{\sum_{i=1}^4 e_i\cdot d(f_\gamma^{-1}\widehat{\psi})(e_i) \\
&\quad + \frac{1}{2}\sum_{i=1}^4\left[(s_Q^*\omega_A + 2f_\gamma^{-1}df_\gamma)(e_i)e_i + \sum_{j<k}\omega_{kj}^U(e_i)e_je_k\right]\cdot(f_\gamma^{-1}\psi)\Bigg\} \\
&\quad - f_\gamma^{-1}\left\{\sum_{i=1}^4 e_i\cdot d\widehat{\psi}(e_i) + \frac{1}{2}\sum_{i=1}^4\left[s_Q^*\omega_A(e_i)e_i + \sum_{j<k}\omega_{kj}^U(e_i)e_je_k\right]\cdot\psi\right\} \\
&= \sum_{i=1}^4 e_i\cdot d(f_\gamma^{-1})(e_i)\psi + \frac{1}{2}\sum_{i=1}^4\left[2f_\gamma^{-1}df_\gamma(e_i)e_i\right]\cdot(f_\gamma^{-1}\psi) \\
&= 0
\end{aligned}$$

as desired. □

This shows that the second of the SW equations is invariant under changes of gauge. What about the first equation? The curvature transforms as

$$F_{\det \gamma^* A} = f_\gamma^{-1} F_A f_\gamma = F_A$$

so $F^+_{\det \gamma^* A} = F^+_A$. As for the part involving q

$$q(f_\gamma^{-1}\psi)\varphi = \left\langle f_\gamma^{-1}\psi | \varphi \right\rangle \psi - \frac{1}{2}\left\langle f_\gamma^{-1}\psi | f_\gamma^{-1} \right\rangle \varphi = |f_\gamma|^{-2} q(\psi)\varphi = q(\psi)\varphi$$

for all $\psi, \varphi \in \Gamma(S^+)$ and $\gamma \in \mathcal{G}$. It follows that the first SW equation is also invariant under gauge transformations. We can tentatively define the *Seiberg-Witten moduli space* to be

$$\mathcal{M}_\sigma := \mathcal{S}_\sigma / \mathcal{G}_\sigma$$

where $\mathcal{S}_\sigma \subset \mathcal{C}_\sigma$ is the space of solutions to the SW equations. If only this action were smooth, there would be hope for saying nice things about this moduli space. Alas, we are sunk immediately as $\mathcal{G}$ is not even a Lie group, so it does not make sense to talk about a smooth $\mathcal{G}$-action on a manifold. This is where Sobolev spaces first come into the picture; which we do not get into here. There is also no hope of having a well behaved moduli space if the solutions are reducible, i.e., $\psi \equiv 0$. This is where the perturbation comes in, but since I'm not getting into that just suppose I'm either including a perturbation or I'm looking at irreducible things.

Theorem 2.2. *$\mathcal{M}_\sigma$ is a smooth,* **compact**, *oriented manifold of dimension*

$$d(\sigma) = \frac{1}{4}(c_1(Q)^2 - 2(\chi(M) + \tau(M))).$$

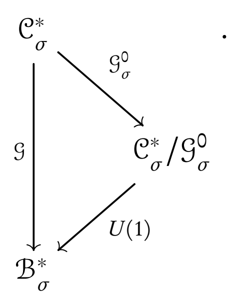
Here $\tau = b_2^+ - b_2^-$.

Actually, I lied to you. One needs to first perturb the equations a bit to get everything to work out. Don't worry about it too much now. Compactness comes from very beautiful curvature identities and bounds involving the Dirac operator.

We need to get our hands on a particular line bundle over $\mathcal{M}_\sigma$. Fix $x_0 \in M$. Let us consider the short exact sequence of groups

$$1 \longrightarrow \mathcal{G}^0_\sigma \longrightarrow \mathcal{G}_\sigma \longrightarrow \mathrm{U}(1) \longrightarrow 1$$

where ev_{x_0} denotes the evaluation map $\gamma \mapsto \gamma(x_0) \in \mathrm{U}(1)$ and where $\mathcal{G}^0_\sigma := \ker\ \mathrm{ev}_{x_0}$. This sequence corresponds to the sequence of principal bundles



Denote by $\mathcal{U}_\sigma$ the complex line bundle associated, by the standard action of S^1 on $\mathbf{C}$, to the principal $U(1)$-bundle $\mathcal{C}^*_\sigma/\mathcal{G}^0_\sigma$ displayed above. To define the Seiberg-Witten invariant we must split up into different cases. Recall that we are restricting to manifolds with $b_2^+ > 1$.

Let's define the invariants. I didn't talk much about the orientation, but when $\dim \mathcal{M}_\sigma = 0$, the SW invariant is defined to be the sum of ± 1 over the points of $\mathcal{M}_\sigma$, where the sign depends somehow on the orientation. When $\dim \mathcal{M}_\sigma > 0$ we define

$$\mathsf{sw}_\sigma(M) = \int_{\mathcal{M}_\sigma} (1 - c_1(\mathcal{U}_\sigma))^{-1}.$$

Here $(1-t)^{-1} = 1 + t + t^2 + \cdots$, which does indeed terminate.

Twisting $\mathcal{N} = 2$, $d = 4$ theory

It is a result of Witten that there is a TQFT whose correlation function are exactly the Donaldson invariants. I would like to briefly give that story here, but I am very unclear of the details. One starts with $\mathcal{N} = 2$, $d = 4$ SYM theory on, say, Euclidean space. One can perform all these manipulations on a general four-manifold, but I want to get the local formulae across.

The data for $\mathcal{N} = 2$, $d = 4$ SYM. We use Euclidean signature. We take our symmetry group to be

$$V \oplus (S_+ \oplus W \oplus S_- \oplus W^*)[1]$$

where V is the defining four dimensional rep of $SO(4)$ and W is two-dimensional. The group of R-symmetries is $G_R = U(2) \cong SU(2) \times U(1)$. In the decomposition $\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_- \times \mathrm{SU}(2) \times \mathrm{U}(1)$ the physical fields are:

$$\begin{aligned}
\text{Gauge field}: \quad & A \in () \\
\text{Spinors (i} = 1,2): \quad & \lambda^i \in \left(\frac{1}{2}, 0, \frac{1}{2}, -1\right) \\
& \overline{\lambda}_i \in \left(0, \frac{1}{2}, \frac{1}{2}, +1\right) \\
\text{Scalars}: \quad & \varphi \in (0,0,0,+2) \ , \ \overline{\varphi} \in (0,0,0,-2).
\end{aligned}$$

And the SUSY fields are:

$$Q_i \in \left(\frac{1}{2}, 0, \frac{1}{2}, +1\right) \quad , \quad \overline{Q}_i \in \left(0, \frac{1}{2}, \frac{1}{2}, -1\right) \quad , \quad i = 1,2.$$

The twist we choose is

$$\rho : \mathrm{Spin}(4) \to \mathrm{SU}(2) \times \mathrm{U}(1) \ , \ (g,h) \mapsto (gh^{-1}, 1).$$