Constructing TQFTs from Categories of Sheaves

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1 Introduction

In this talk, I'll try to synthesise some of the ideas from the earlier talks in order to describe an example of a fully extended 2d TQFT. This TQFT will be built from geometric data following the "bottom-up" approach of the cobordism hypothesis. That is, we'll choose a category of geometric objects (quashicoherent sheaves on a stack) to assign to the point, and explore some of the consequences of this assignment at the level of higher-dimensional manifolds. We'll be following the aproach of the paper "Integral Transforms and Drinfeld Centers in Derived Algebraic Geometry" by Ben-Zvi, Francis and Nadler [1]. The main theorem that will help us to understand this example is an identification of the category of quasicoherent sheaves on a fibre product in terms of the tensor product of the categories of sheaves on the factors, or in terms of the category of functors between these categories

$$QC(X_1 \times_Y X_2) \cong QC(X_1) \otimes_{QC(Y)} QC(X_2) \cong \mathrm{Fun}_{QC(Y)}(QC(X_1), QC(X_2)).$$

In particular, we see that the Hochschild homology and cohomology of the category $QC(X)$ are both identified with $QC(\mathcal{L}X)$, the category of sheaves on the loop space.

2 2d Topological Quantum Field Theories

Let's set up the kind of objects we'll be constructing, and in doing so, recall some of the ideas from Pavel's talk yesterday. We'll construct a *two-dimensional, fully extended, oriented, categorified TQFT.* All these adjectives need some unpacking. A fully extended 2d TQFT is, as ever, a functor

$$Z \colon 2\,\mathrm{Bord} \to \mathcal{C}$$

where $2\,\mathrm{Bord}$ is an $(\infty, 2)$-category of *bordisms*, and $\mathcal{C}$ is a target $(\infty, 2)$-category. These categories should be symmetric monoidal, and the functor Z should be a symmetric monoidal functor.

We'll work with the *oriented* bordism category (though in fact the functor we will construct also defines an unoriented theory). This is worth remarking from the point of view of the cobordism hypothesis (see [3]): the simplest form of the cobordism hypothesis constructs a *framed* TQFT from a fully dualisable object of $\mathcal{C}$. Being part of an unoriented TQFT imposes more restrictions on this object. The bordism category has a symmetric monoidal structure coming from disjoint union.

For our target category, we'll use a *categorified Morita category* of algebras: that is, a Morita category internal to ∞-categories. To spell this out a bit more we have

- Objects: algebra objects in ∞-categories.
- 1-Morphisms: bimodule objects in ∞-categories.
- 2-Morphisms and higher: (the classifying space of) morphisms of bimodule objects.

When I say ∞-category in this talk, I'll always mean objects of the category $\mathcal{S}t^L$ of stable, presentable ∞-categories with morphisms colimit-preserving functors. This category has a symmetric monoidal structure by tensor product (of presentable ∞-categories), as Patrick explained on Monday.

Suppose we have such a TQFT, with $Z(\mathrm{pt}) = A$. This imposes some fairly strong conditions on A, as we saw yesterday. For a start (for A to correspond to at least a *framed* theory), A must be *fully dualisable*. It must have a dual *object*, and the evaluation 1-morphism must have right and left adjoints. If we consider the circle with its trivial 2-framing (coming from a 1-framing), we can split it into two semicircles (evaluation and coevaluation), and observe that

$$Z(S^1) = A \otimes_{A \otimes A^{op}} A = HH_\bullet(A)$$

the *Hochschild homology* of A. As we saw in Lee's talk, for A to correspond to an oriented theory we further require A to be invariant under an $SO(2)$ action. In particular, the Hochschild homology must be isomorphic to the *Hochschild cohomology* of A, which is assigned to the circle with the framing coming from the boundary of a disc (we replace the semicircle coev by the left adjoint ev^L, which is the composition of coev with a Serre automorphism. In an oriented theory the Serre automorphism is trivialised as Pavel explained, which is just a fancy way of saying that the induced orientated circles coming from these two circles are related by an oriented diffeomorphism. The existence of the TQFT imposes this condition on A. There is a detailed discussion of extended 2d TQFTs explaining these issues in [2] chapter 2.

Furthermore, this Hochschild (co)homology object must come with further structure, coming from the 2d bordisms in the source category. A bordism

$$\coprod_m S^1 \to \coprod_n S^1$$

induces a homomorphism

$$Z(S^1)^{\otimes m} \to Z(S^1)^{\otimes n}$$

in a way compatible with composition. In other words, the Hochschild (co)homology $HH_\bullet(A)$ gains the structure of an E_2-algebra. That is, we can view a configuration of m circles in a disk as a 2-cobordism $(S^1)^{\sqcup m} \to S^1$, so applying the functor Z to this 2-morphism we get a map of classifying spaces

$$\mathrm{Conf}_m \to \mathrm{Hom}(HH_\bullet(A)^{\otimes m}, HH_\bullet(A))$$

where Conf_m denotes the relevant configuration space of circles.

3 2d TQFTs from Sheaves on a Stack

We're going to, following Ben-Zvi–Francis–Nadler, construct an example of such a TQFT. We'll try to work in a "bottom-up" fashion, starting with the algebra object assigned to the point, and working up to the bimodules and homomorphisms assigned to 1- and 2-manifolds.

Let X be a *perfect stack*, in the sense of Justin's talk earlier today. For instance, X might be a quasi-compact scheme with affine diagonal, or a quotient stack of the form Y/G, where G is an affine algebraic group acting on a variety Y. Later on we'll consider the specific example $X = BG$, for G an affine algebraic group. When G is finite, this recovers the (Dijkgraaf-Witten) finite group gauge theory discussed yesterday. I'll avoid using any details from the language of derived stacks beyond the most formal calculations.

We have the (stable, presentable) ∞-category $QC(X)$ of quasi-coherent sheaves on X. This is a monoidal ∞-category via the tensor product of $\mathcal{O}_X$-modules. From now on, lets work relative to a (derived if we like) commutative ring k. So X is a stack over k, and $QC(X)$ is a k-algebra object in ∞-categories.

Now, we'd like to construct a TQFT from this category in the manner described above. To do this, we need $QC(X)$ to be *self-dual* as an algebra. That is, we need maps (unit and trace)

$$1 \xrightarrow{u} QC(X) \otimes QC(X) \xrightarrow{t} 1$$

where 1 is the monoidal unit in k-linear ∞-categories, namely $\mod_k$, such that the composite

$$QC(X) \xrightarrow{u \otimes 1} QC(X) \otimes QC(X) \otimes QC(X) \xrightarrow{1 \otimes t} QC(X)$$

is the identity. Furthermore, we need to know that $QC(X)$ is $O(2)$-invariant, and in particular that

$$HH_\bullet(QC(X)) \cong HH^\bullet(QC(X)).$$

These are both consequences of the following theorem, computing quasicoherent sheaves on a fibre product:

Theorem 3.1 ([1] 4.7, 4.10)**.** Let $p_1 \colon X_1 \to Y$, $p_2 \colon X_2 \to Y$ be maps of perfect stacks. Then there are canonical equivalences

$$\begin{aligned} QC(X_1 \times_Y X_2) &\cong QC(X_1) \otimes_{QC(Y)} QC(X_2) \\ &\cong \mathrm{Fun}_{QC(Y)}(QC(X_1), QC(X_2)) \end{aligned}$$

Note that we're working in a derived setting throughout, so we always mean the *derived* fibre product. We'll sketch this theorem towards the end, but first we'll derive some consequences, and finish constructing the TQFT. Firstly:

Corollary 3.2. If X is a perfect stack, $QC(X)$ is self-dual.

Proof. We have $QC(X) \otimes QC(X) \cong QC(X \times X)$, so we can produce the unit and trace by $u = \Delta_* \pi^*$, $t = \pi_* \Delta^*$, where Δ is the diagonal embedding and $\pi \colon X \to \operatorname{Spec} k$. So the unit sends a k-vector space to a power of the structure sheaf of the diagonal, and the trace computes the cohomology of the restriction of a sheaf to the diagonal. To check that this really does exhibit $QC(X)$ as self-dual is a base-change argument ([1] 4.9).

We can see a relative version as well, i.e. that $QC(X)$ is self-dual as a $QC(Y)$-module, given a map $\pi \colon X \to Y$. The unit and trace are given by exactly the same functors as before, where Δ is now the diagonal $X \to X \times_Y X$. To see that this exhibits $QC(X)$ as self-dual, we note that $u \otimes 1$ corresponds to the map $(\Delta_{12} \otimes 1)_*(\pi_1)^*$, and $1 \otimes t$ corresponds to the map $(\pi_2)_*(1 \otimes \Delta_{23})^*$. So we can apply base change to the commutative diagram

$$\begin{tikzcd} & X \arrow[r, "\Delta"] \arrow[d, "\Delta"] & X \times_Y X \arrow[r, "\pi_1"] \arrow[d, "1\otimes\Delta_{23}"] & X , \\ X & X \times_Y X \arrow[l, "\pi_2"'] \arrow[r, "\Delta_{12}\times 1"] & X \times_Y X \times_Y X & \end{tikzcd}$$

and note that $\pi_i \circ \Delta = 1 \colon X \to X$.

□

Furthermore, we can understand the Hochschild homology and cohomology from this point of view, i.e. the object the TQFT assigns to the circle. By the theorem, we have

$$\begin{aligned} HH^\bullet(QC(X)) &= \mathrm{Fun}_{QC(X)\otimes QC(X)}(QC(X), QC(X)) \\ &= QC(X \times_{X \times X} X) \\ &= QC(X) \otimes_{QC(X)\otimes QC(X)} QC(X) = HH_\bullet(QC(X)). \end{aligned}$$

What exactly is this object? We have $X \times_{X \times X} X$ where the maps are both the diagonal embedding. This space is also known as the *derived (free) loop space* $\mathcal{L}X$, or in other words the *mapping space* X^{S^1}. Indeed, from a topological point of view, we can compute this homotopy fibre product by replacing the space X by the homotopy equivalent path space $\mathcal{P}X$, and the diagonal map $X \to X \times X$, by the map $\mathcal{P}X \to X \times$ sending a path to its end points. From this point of view, points in the fibre product $\mathcal{P}X \times_{X \times X} \mathcal{P}X$ correspond to pairs of paths with the same end points, i.e. closed loops in X. Thus we have shown that the Hochschild homology and cohomology of $QC(X)$ agree, and are given by

$$QC(\mathcal{L}X).$$

We conclude our description of the TQFT by describing the maps coming from a bordism $\Sigma \colon \coprod_m S^1 \to \coprod_n S^1$. By the above discussion, we have seen $Z(\coprod_m S^1) = QC(\mathcal{L}X^m)$. This suggests natural maps associated to bordisms coming from pushing and pulling sheaves. That is, the inclusion of the incoming and outgoing boundary of Σ induces a diagram

$$\begin{tikzcd} & X^\Sigma \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\ (\mathcal{L}X)^m & & (\mathcal{L}X)^n \end{tikzcd}$$

and hence a functor $(f_2)_*(f_1)^* \colon QC(\mathcal{L}X^m) \to QC(\mathcal{L}X^n)$. These functors exhibit the E_2-algebra structure on $QC(\mathcal{L}X)$.

Of course, this doesn't give a full description of a 2d fully extended TQFT. There are still a lot more pieces of data to describe, and compatibilities to check. However, we have at least described all of the data in a 2d non-extended TQFT. To check that this describes a functor, it is just necessary to check compatibility with composition. That is, we consider diagrams given by sewing bordisms, of the form

$$\begin{tikzcd} & & X^\Sigma \arrow[dl] \arrow[dr] & & \\ & X^{\Sigma_1} \arrow[dl] \arrow[dr] & & X^{\Sigma_2} \arrow[dl] \arrow[dr] & \\ (\mathcal{L}X)^{m_1} & & (\mathcal{L}X)^{m_2} & & (\mathcal{L}X)^{m_3} \end{tikzcd}$$

and use base change to check the appropriate composition identity holds for the maps of sheaves.

Example 3.3. Let's consider what form this theory takes in the example $X = BG$, for G an affine algebraic group over a field k of characteristic zero (for simplicity). BG is an example of a perfect stack, so we can apply the above construction to produce a TQFT that assigns $QC(BG)$ to the point. We might interpret this as a *derived category of representations of* G. Indeed, suppose that G is a finite group, and consider the analogous underived category. Then a quasicoherent sheaf on BG is just the same as a k-vector space with a (right, say) G-action, i.e. a right kG-module.

Our TQFT assigns to the circle the category $QC(\mathcal{L}BG)$. The stack $\mathcal{L}BG$ is equivalent to the *adjoint quotient of* G, i.e. the quotient G/G where G acts on itself by conjugation. We can give a topological argument for this fact as follows: a point in $\mathcal{L}BG$ is the same as a map $S^1 \to BG$, which corresponds to a principal G-bundle on S^1. We can produce all principal G-bundles on S^1 by taking the trivial G-bundle on the interval $[0, 1]$, and gluing the fibres at the end points 0 and 1 together. If we trivialise the fibre at 0, then this gluing map corresponds to multiplication by an element $g \in G$, and changing this trivialisation corresponds to conjugating this element g, thus we can identify such a G-bundle with a point in the adjoint quotient G/G.

What about the maps coming from 2-manifolds? Let Σ be a closed 2-manifold. Then we are considering the correspondence

$$\begin{tikzcd} & BG^\Sigma \arrow[dl] \arrow[dr] & \\ \varnothing & & \varnothing \end{tikzcd},$$

that is, the functor $\mathrm{Mod}_k \to \mathrm{Mod}_k$ that sends the generator k to $H^*(BG^\Sigma; \mathcal{O}_{BG^\Sigma})$. The mapping space BG^Σ is also known as the moduli stack of G-local systems $\mathrm{Loc}_G(\Sigma)$ (where G is given the *discrete* topology, so bundles and local systems coincide), and our TQFT computes the cohomology of this moduli stack.

4 Sheaves on Fibre Products

Now we'll explain where Theorem 3.1 comes from, and why we need the hypothesis that the underlying derived stacks are perfect.

First, let's compare $QC(X_1 \times X_2)$ and $QC(X_1) \otimes QC(X_2)$. There is a natural functor between them, namely the *external product*

$$\boxtimes \colon QC(X_1) \otimes QC(X_2) \to QC(X_1 \times X_2).$$

In the case when X_i are *perfect*, it suffices to investigate this functor on compact objects. Since compact and perfect objects coincide for a perfect stack, these objects generate the whole category.

Proposition 4.1. The above functor defines an equivalence on the level of compact objects:

$$\boxtimes\colon QC(X_1)^c \otimes QC(X_2)^c \xrightarrow{\sim} QC(X_1 \times X_2)^c.$$

Proof, following [1]. The crucial fact is that this functor is essentially surjective, i.e. that $QC(X_1 \times X_2)$ is generated by external products of compact objects. We argue this as follows. Suppose $\mathrm{Hom}(M_1 \boxtimes M_2, N) \cong 0$ for all $M_i \in QC(X_i)^c$. We can deduce that $N \cong 0$ following [1] Proposition 3.24. Unpacking this, we have

$$\begin{aligned}\mathrm{Hom}(M_1 \boxtimes M_2, N) &= \mathrm{Hom}(\pi_1^* M_1, \mathcal{H}om(\pi_2^* M_2, N)) \\ &= \mathrm{Hom}(M_1, \pi_{1*}\mathcal{H}om(\pi_2^* M_2, N)).\end{aligned}$$

The M_1 generate, so the sheaf on X_1, $\pi_{1*}\mathcal{H}om(\pi_2^* M_2, N) \cong 0$. We analyse this sheaf now on an affine open $U \to X_1$.

$$\begin{aligned}\Gamma(U, \pi_{1*}\mathcal{H}om(\pi_2^* M_2, N) \cong 0) &= \mathrm{Hom}_{U \times X_2}(\pi_2^* M_2, N) \\ &= \mathrm{Hom}_{X_2}(M_2, (\pi_2|_{U \times X_2})_* N).\end{aligned}$$

The M_2 generate, so the sheaf on X_2, $\pi_2|_{U\times X_2 *} N \cong 0$. Thus for any affine open $V \to X_2$, $\Gamma(U \times V; N) \cong 0$. Such affine opens cover $X_1 \times X_2$, so $N \cong 0$ as required, and the $M_1 \boxtimes M_2$ generate.

Given this, the fact that the functor is fully faithful is a calculation using the projection formula. Indeed, we compute

$$\begin{aligned}\mathrm{Hom}(M_1 \boxtimes M_2, N_1 \boxtimes N_2) &= \Gamma(X; p_1^* M_1^\vee \otimes p_2^* M_2^\vee \otimes p_1^* N_1 \otimes p_2^* N_2) \\ &= \Gamma\left(X; p_1^* \mathcal{H}om_{X_1}(M_1, N_1) \otimes p_2^* \mathcal{H}om_{X_2}(M_2, N_2)\right) \\ &= \Gamma\left(X_2; (p_2)_* \left(p_1^* \mathcal{H}om_{X_1}(M_1, N_1) \otimes p_2^* \mathcal{H}om_{X_2}(M_2, N_2)\right)\right)\end{aligned}$$

where this calculation relies upon the fact that the M_i are fully dualisable. Now apply the projection formula to the morphism p_2.

$$\begin{aligned}\mathrm{Hom}(M_1 \boxtimes M_2, N_1 \boxtimes N_2) &= \Gamma\left(X_2; \mathrm{Hom}_{X_1}(M_1, N_1) \otimes \mathcal{H}om_{X_2}(M_2, N_2)\right) \\ &= \mathrm{Hom}_{X_1}(M_1, N_2) \otimes \mathrm{Hom}_{X_2}(M_2, N_2)\end{aligned}$$

as required.

□

Now, let's go on to consider the case of a general fibre product $X_1 \times_Y X_2$. Again, I'll only sketch the arguments given in [1]. We'll analyse the category $QC(X_1) \otimes_{QC(Y)} QC(X_2)$ by taking a "resolution", i.e. viewing it as the colimit of a simplicial diagram in ∞-categories. There is a standard way of doing this, by taking the "two-sided bar construction", which we already saw a version of in Shilin's talk yesterday, which allowed us to compute a derived tensor product (as Hochschild homology groups for a bimodule are Tor groups). So we take the colimit (geometric resolution) of the simplicial diagram

$$QC(X_1) \otimes QC(X_2) \leftleftarrows QC(X_1) \otimes QC(Y) \otimes QC(X_2) \,\substack{\longleftarrow\\[-1pt]\longleftarrow\\[-1pt]\longleftarrow}\, QC(X_1) \otimes QC(Y) \otimes QC(Y) \otimes QC(X_2) \cdots$$

with arrows given by contracting tensors, and produce a category equivalent to $QC(X_1) \otimes_{QC(Y)} QC(X_2)$. Alternatively, we could've taken the *limit* (totalization) of the *cosimplicial* diagram

$$QC(X_1) \otimes QC(X_2) \rightrightarrows QC(X_1) \otimes QC(Y) \otimes QC(X_2) \,\substack{\longrightarrow\\[-1pt]\longrightarrow\\[-1pt]\longrightarrow}\, QC(X_1) \otimes QC(Y) \otimes QC(Y) \otimes QC(X_2) \cdots.$$

with arrows given by the right adjoints of the maps in the previous diagram. This limit is just the previous colimit evaluated as a limit in the *opposite* category $\mathcal{S}t^R$ to $\mathcal{S}t^L$.

In both cases, we can apply the previous theorem 4.1 for the absolute case to interpret these diagrams in a geometric way. The diagrams are both induced from the cosimplicial diagram of stacks

$$X_1 \times X_2 \rightrightarrows X_1 \times Y \times X_2 \,\substack{\longrightarrow\\[-1pt]\longrightarrow\\[-1pt]\longrightarrow}\, X_1 \times Y \times Y \times X_2 \cdots$$

with arrows given by pullback along the maps in the first case, and pushforward in the second case.

Ok, so this gives us a lot of different ways of viewing the category $QC(X_1) \otimes_{QC(Y)} QC(X_2)$. How does it help us? Well, we now have a lot of maps to work with, allowing us to compare this category to other categories of a more geometric origin. First of all, we have an adjunction

$$QC(X_1) \otimes_{QC(Y)} QC(X_2) \underset{\tau^*}{\overset{\tau_*}{\rightleftarrows}} QC(X_1 \times X_2)$$

by looking at purely the 0-(co)simplices in our resolutions: inclusion of the 0-simplices, or a map from the totalization to the 0-cosimplices. Secondly, we have another easy adjunction, coming from the universal map $\pi \colon X_1 \times_Y X_2 \to X_1 \times X_2$, namely

$$QC(X_1 \times_Y X_2) \underset{\pi^*}{\overset{\pi_*}{\rightleftarrows}} QC(X_1 \times X_2) \ .$$

In other words, we've described a pair of *monads* on the category $QC(X_1 \times X_2)$.

Now, it seems natural to try to apply the monadicity theorem. Indeed, one can check that both the maps τ_* and π_* are conservative and preserve all colimits, so our two categories $QC(X_1) \otimes_{QC(Y)} QC(X_2)$ and $QC(X_1 \times_Y X_2)$ are categories of modules for monads on $QC(X_1 \times X_2)$.

The proof concludes with the most important step: these monads are actually *equivalent.* We construct maps between them by *factoring* the maps π_* and π^* *through* the category $QC(X_1) \otimes_{QC(Y)} QC(X_2)$. There's one more natural adjunction that we haven't yet considered. The map π doesn't just induce maps between $QC(X_1 \times_Y X_2)$ and $QC(X_1 \times X_2)$. It actually induces maps to and from the whole (co)simplicial category $QC(X_1) \otimes_{QC(Y)} QC(X_2)$. That is, there is an adjunction

$$QC(X_1) \otimes_{QC(Y)} QC(X_2) \underset{\widetilde{\pi}_*}{\overset{\widetilde{\pi}^*}{\rightleftarrows}} QC(X_1 \times_Y X_2) \ .$$

We can view π^* for instance as taking a sheaf on $QC(X_1 \times X_2)$, including the 0-simplices into the whole simplicial ∞-category representing the tensor product, and then pulling back to a sheaf $X_1 \times_Y X_2$ along the augmentation π. In other words, $\pi^* = \widetilde{\pi}^* \tau^*$, and similarly $\pi_* = \tau_* \widetilde{\pi}_*$. Thus we have maps of monads

$$\tau_* \tau^* \rightleftarrows \pi_* \pi^* = \tau_* \widetilde{\pi}_* \widetilde{\pi}^* \tau^*$$

given by the unit and counit of the $\widetilde{\pi}$ adjunction. One can check that in fact these maps define an equivalence, and thus the algebraic and geometric categories coincide, which proves the first equivalence of Theorem 3.1.

The self-duality of $QC(X)$ as a $QC(Y)$-module produces the second equivalence immediately, since

$$\mathrm{Fun}_{QC(Y)}(QC(X_1), QC(X_2)) \cong QC(X_1)^\vee \otimes_{QC(Y)} QC(X_2) \cong QC(X_1) \otimes_{QC(Y)} QC(X_2).$$

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