Symmetries in Conformal Field Theory

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These are elementary notes on Virasoro and affine Lie algebra symmetries in 2d conformal field theory, prepared for a seminar talk at Northwestern. In preparing these notes I referred to notes of Tong [Ton] and Ginsparg [Gin89], as well as the book [FS10] of Frischman and Sonnenschein.

1 Action of the Virasoro Algebra

1.1 Positive Energy Representations

Let's begin by recalling a fact about 2d conformal field theories from Phil's talk last week. Firstly, we can define a complex semigroup $\mathcal{A}$ whose objects are conformal equivalence classes of Riemann surfaces which are diffeomorphic to annuli, equipped with parameterisations of their boundary components, one incoming and one outgoing. The semigroup operation is given by gluing an outgoing to an incoming boundary component. This semigroup is isomorphic to

$$\left((0,1) \times \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)\right)/U(1)$$

where $\mathrm{Diff}^+(S^1)$ is the monoid of orientation preserving diffeomorphisms of the circle, and where the semigroup operation on $(0,1)$ is multiplication. Indeed any object of $\mathcal{A}$ is conformally equivalent to an annulus in the plane, and two such are equivalent if the ratios of the radii of the inner and outer circles coincide. The $\mathrm{Diff}^+(S^1)$ factors describe the parameterisations of the boundary circles.

Proposition 1.1. There is a bijection between *holomorphic* projective representations of $\mathcal{A}$ and *positive energy* projective representations of $\mathrm{Diff}^+(S^1)$.

As such, we think of $\mathcal{A}$ as being a kind of "complexification" of the semigroup $\mathrm{Diff}^+(S^1)$. For this to make sense, we should say what we mean when we say "positive energy". This makes sense for any group G with a circle action, for instance $\mathrm{Diff}^+(S^1)$ with the action of rigid rotations, or the loop group LG of a Lie group.

Definition 1.2. A representation V of $U(1)$ called *positive* if it can be decomposed into a linear combination of characters $e^{ik\theta}$ where $k > 0$. Let G be a group with a circle action. A projective representation V of G is called *positive energy* if it extends to a projective representation of the semidirect product $U(1) \ltimes G$ where $U(1)$ acts by a positive representation.

In the case of $\mathrm{Diff}^+(S^1)$ this is just saying that the subgroup of rigid rotations acts positively. We'll come back to the notion of positive energy later, from a more physical perspective.

1.2 The Stress-Energy Tensor

Recall that a *current* for a classical field theory in n-dimensions is a local assignment of an $(n-1)$-form to every field, defined up to the addition of an exact $(n-1)$-form. Its associated *charges* are the functionals obtained by integrating the current along a compact oriented $(n-1)$-dimensional submanifold. The current is called *conserved* if it lands in the subspace of *closed* forms, so is given by a map to $H^{n-1}(X)$. In particular this means that charges are independent of the homology class of the codimension 1 submanifold.

We'll use the following fundamental theorem from classical field theory

Theorem 1.3 (Noether's Theorem)**.** There is an equivalence between infinitesimal symmetries of a classical field theory and conserved currents.

Remark 1.4. A modern formulation of this theorem appears as theorem 5.16.2.1 in [CG14].

Specifically, we'll be interested in applying this theorem to a 2d conformal field theory, since we'll obtain conserved currents from the full group of conformal symmetries. These will comprise the (local) *stress-energy tensor* of the theory. More specifically, we can view any sufficiently small conformal transformation as a deformation of the metric

$$g_{ab} \mapsto g_{ab} + \varepsilon h_{ab},$$

where h_{ab} is a symmetric 2-tensor. So there's an infinitesimal symmetry associated to every such h_{ab}. We denote – in coordinates – the corresponding conserved currents by $T_{ab}(p)$, which is a conserved current depending on values for a, b and a point p in spacetime. In complex local coordinates $z, \overline{z}$ for a Riemann surface we write T for the conserved current T_{zz}. Similarly, we define $\overline{T}$ to be the conserved current $T_{\overline{z}\overline{z}}$.

So we've produced some classical observables from the conformal symmetries. Let's imagine that we had a consistent quantisation for some conformal field theory, and investigate what properties the corresponding *quantum observables* ought to satisfy. From now on we'll specify to the case where spacetime is described by an annulus $A \subseteq \mathbb{C}^\times$. Our quantum observables are holomorphic operator-valued currents, so decompose as a Laurent series

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-(n+2)}$$

where the odd normalisation comes from a natural isomorphism $\mathbb{C}^\times \cong S^1 \times \mathbb{R}$ and is standard in the physics literature. We'll investigate the Laurent components L_n, in particular their commutation relations. They can be computed by the integral

$$L_n = \frac{1}{2\pi i} \oint T(z) z^{n+1}.$$

Thus

$$[L_n, L_m] = \frac{-1}{4\pi^2} \oint \oint (z^{n+1} w^{m+1} - z^{m+1} w^{n+1}) T(z) \cdot T(w),$$

where the observable $T(z) \cdot T(w)$ is the *operator product* of $T(z)$ and $T(w)$: as an observable on $A \times A$ it's perfectly well defined *except* along the diagonal, where it diverges. In general such operator products admit operator product *expansions* (OPEs), which on the annulus take the form

$$\mathcal{O}(z)\mathcal{O}'(w) = \sum_{k=-\infty}^{\infty} \mathcal{O}_k(w)(z-w)^k$$

for some sequence of observables $\mathcal{O}_k$. One can show that

$$T(z) \cdot T(w) = \frac{c/2}{(z-w)^4} + \frac{T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots$$

for some constant c depending on the theory in question, called the *central charge* or *Virasoro anomaly* of the theory. Broadly speaking, one computes this by showing that derivatives of $T(w)$ form a complete basis of local observables in the theory, and computing the OPE in the theory describing a free boson where one can write out the stress energy tensor explicitly. This OPE calculation allows us to do the integral computing $[L_n, L_m]$ with a little care. One interprets the z integral as computing the residue at $z = w$, which we can do by expanding as a

Laurent series in z about w. The result is

$$\begin{aligned}[L_n, L_m] &= \frac{1}{2\pi i}\oint w^{n+1}(w^{m+1}\partial_w T(w) + 2(m+1)w^m T(w) + \frac{c}{12}m(m^2-1)w^{m-2})dw \\ &= -(n+m+2)L_{n+m} + 2(m+1)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{n,m} \\ &= (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{n,m}.\end{aligned}$$

The anti-holomorphic operator $\overline{T}$ has components $\overline{L}_n$ satisfying similar commutation relations, but with a possibly different central charge $\widetilde{c}$.

This motivates the following definition

Definition 1.5. The *Virasoro algebra* is the Lie algebra generated by elements L_n for $n \in \mathbb{Z}$ and a central element c with commutation relations

$$[L_m, L_m] = (m-n)L_{n+m} + \frac{c}{12}m(m^2-1)\delta_{n,m}.$$

We've just proven that the Hilbert space in a 2d conformal field theory carries a canonical representation of the Virasoro algebra coming from the Laurent coefficients of the stress-energy tensor.

1.3 The Virasoro Algebra from $\mathrm{Diff}^+(S^1)$

There's another, equivalent way of obtaining this canonical action. We can decompose an element $f \in \mathrm{Diff}^+(S^1)$ into Fourier modes

$$f(\theta) = \sum_{n=-\infty}^{\infty} \alpha_n(f)e^{i\pi\theta}.$$

On the level of the Lie algebra of the symmetry group, we obtain a generator ℓ_n for each Fourier coefficient (generating one parameter families of deformations of the identity diffeomorphism). Classically these symmetry generators act by the vector fields $z^{n+1}\partial_z$ on the annulus, which one observes satisfy the commutation relations

$$[\ell_n, \ell_m] = (m-n)\ell_{n+m}.$$

This is all well and good classically, but in the quantum theory, there's a correction term controlled by the Virasoro anomaly. This is related to the fact that the Hilbert space representation quantising these symmetries is only projectively defined. The operators L_n quantise the Fourier components ℓ_n, generating symmetries of the classical theory.

1.4 Representations of the Virasoro Algebra

We'll conclude this section with a brief discussion of the representation theory of the Virasoro algebra, in particular we'll discuss certain natural representations called *highest weight representations*. These representations will have a natural physical interpretation. Suppose we have some state $|\phi\rangle$ in the Hilbert space which is a simultaneous eigenstate for the operators L_0 and $\overline{L}_0$ with eigenvalue h and $\widetilde{h}$ respectively. Then the state $L_n|\phi\rangle$ is also an eigenstate for these operators with eigenvalue $h-n$ and $\widetilde{h}$ respectively, since L_n commutes with $\overline{L}_0$, and

$$L_0 L_n|\phi\rangle = (L_n L_0 - nL_n)|\phi\rangle = (h-n)L_n|\phi\rangle.$$

The pair of eigenvalues $h, \widetilde{h}$ encodes the *energy* of the state $|\phi\rangle$. Indeed, viewing the annulus as the worldsheet of a string, with the longitudinal direction thought of as time, we find the time translation operator (the Hamiltonian) is given by

$$H = L_0 + \widetilde{L}_0 - \frac{c+\widetilde{c}}{24}$$

(where the factors of c and $\widetilde{c}$ came from the transformation of the stress energy tensor under a coordinate change from a planar annulus to a cylinder). Thus the energy of the eigenstate $|\phi\rangle$ is exactly $h + \widetilde{h} - \frac{c+\widetilde{c}}{24}$. In order for the possible energies of states to be bounded below, as we'd desire in a physically realistic system, there must exist some state which is annihilated by all L_n and $\overline{L}_n$ for $n > 0$. Such a state is called a state of *highest weight*.

Associated to an eigenvalue h we can build an irreducible *highest weight representation* of the Virasoro algebra, which we should think of as the representation freely generated by acting on a single highest weight state.

Definition 1.6. The *highest weight representation* of the Virasoro algebra of weight $h \geq 0$ and central character $C \geq 1$ is the unique irreducible unitary representation [1] V (up to isomorphism) containing a vector v such that $cv = Cv$, $L_0 v = hv$ and $L_n v = 0$ for $n > 0$. It admits an eigenspace decomposition for L_0 of form

$$V = \bigoplus_{k \geq 0} V_{h+k}$$

where the $h+k$ eigenspace V_{h+k} is spanned by vectors of the form

$$L_{-n_j} \cdots L_{-n_1} v \text{ where } 0 < n_1 \leq \cdots \leq n_j \text{ and } n_1 + \cdots + n_j = k.$$

Remark 1.7. The conditions on h and C are necessary for the highest weight representation (which always exists) to be unitary. We might interpret the first condition as a *positive energy* condition, and the second as a condition on the Virasoro anomaly in the theory. To see where these come from we compute that

$$\begin{aligned}\|L_{-n} v\|^2 &= \langle v, [L_n, L_{-n}] v \rangle \\ &= \left(2nh + \frac{c}{12} n(n^2 - 1)\right) \|v\|^2\end{aligned}$$

using the unitarity condition $L_n^\dagger = L_{-n}$ and considing $n = 1$ and sufficiently large values of n (this actually only shows $c \geq 0$, showing $c \geq 1$ is more subtle). In fact the positive energy condition here is equivalent to the positive energy condition we discussed for representations of $\mathrm{Diff}^+(S^1)$ in the first section.

Thus the Hilbert space in our conformal field theory can always be thought of as a family of such highest weight representations over its space of highest weight states. We won't discuss this further, but there is more to say; conformal field theories admits a *state operator correspondence*, under which the highest weight states correspond to so-called *primary operators* – those local operators $\mathcal{O}$ such that the OPE $\mathcal{O} \cdot \mathcal{O}'$ has at most quadratic poles for all $\mathcal{O}'$. A primary operator has a *conformal weight* (h, h') corresponding to its behavious under the action of ∂ and $\overline{\partial}$ which corresponds to the weight of the corresponding highest weight state.

2 Affine Lie Algebras and the WZW Model

In this second section we'll specialise to conformal field theories with richer families of symmetries, most crucially the *WZW models*. These are 2d conformal field theories with a classical action of a Lie group G. The quantum theories inherit a large symmetry algebra extending both these symmetries and the Virasoro type symmetries we saw in the previous section (in a sense we'll explain). In this section I follow the notes [Gaw99] of Gawędzki fairly closely.

2.1 The WZW Model

Let G be a compact connected simply-connected Lie group. The *WZW (Wess-Zumino-Witten) model* associated to G is a conformal field theory modifying a 2d sigma model with target G. So, from a Lagrangian point of view

[1] By a unitary representation of the Virasoro algebra we mean a homomorphism $\mathrm{Vir} \to \mathfrak{gl}(V)$, where V is equipped with a positive definite Hermitian form, such that the antilinear involution sending L_n to L_{-n} intertwines the Hermitian conjugatation operation on V.

we fix a Riemann surface Σ and define the space of fields in our theory to be the space of smooth maps $\phi\colon \Sigma \to G$, on which G acts by left multiplication. As a first pass, we fix an invariant pairing $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$, and consider the usual sigma model Lagrangian density for a canonical metric on G

$$\mathcal{L}_\Sigma(\phi) = \langle \phi^{-1}\partial\phi, \phi^{-1}\overline{\partial}\phi\rangle$$

where ϕ^{-1} is the function given by post-composing ϕ with the inversion function $G \to G$. This density is conformally invariant, and invariant under the global action of G by left multiplication. However, it turns out the conformal invariance is anomalous when we try to quantise.

We'd like to modify the theory to fix this issue. In particular, invariant of the quantum theory under such transformations will give us an action of the loop group LG on the classical theory which we'll be able to promote to an affine Lie algebra action on the Hilbert space. Witten [Wit84] demonstrated how to do this by adding a "Wess-Zumino" topological term to the action. This will be – roughly speaking – given by the pullback along ϕ of a certain canonical 2-form β on the target, but we'll have to be somewhat sneaky.

We choose a handlebody B, a 3-manifold whose boundary ∂B is isomorphic to Σ. What we'll actually do is to specify a canonical 3-form χ in $\Omega^3(B)$, extend the function ϕ to $\widetilde{\phi}\colon B \to G$, and add to our action by $\int_B \widetilde{\phi}^*\chi$. The 3-form χ will be closed but not exact, so while locally our modification is of the form $\phi^*\beta$, there's no consistent way of gluing these local pieces together.

Definition 2.1. The *canonical 3-form* on a 3-manifold M associated to the group G is given by

$$\chi = \frac{1}{3}\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle.$$

Definition 2.2. The action in the WZW model is given by

$$\begin{aligned} S(\phi) &= \frac{1}{4\pi}(S_\Sigma(\phi) + kS_{\mathrm{WZ}}(\widetilde{\phi})) \\ &= \frac{1}{4\pi}\int_\Sigma \langle \phi^{-1}\partial\phi, \phi^{-1}\overline{\partial}\phi\rangle + \frac{k}{12\pi}\int_B \widetilde{\phi}^*\left(\langle g^{-1}dg, [g^{-1}dg, g^{-1}dg]\rangle\right) \end{aligned}$$

where k is a real constant called the *level.*

There are several comments to be made at this point.

Remarks 2.3.

1. Firstly, note that an extension of $\phi\colon \Sigma \to G$ to a function $\widetilde{\phi}\colon B \to G$ always exists. Indeed, $\pi_i(G)$ is trivial for $i < 3$ ($\pi_2(G)$ is always trivial when G is a Lie group), so all maps $\Sigma \to G$ are nullhomotopic.

2. Nevertheless, the extension is generally *not* unique, which means the action $S(\phi)$ given above is actually not well defined. This is certainly not a problem for the classical theory, because the first variation δS *is* independent of ϕ (we'll discuss this momentarily). However, at the quantum level there might be an issue. However, in the quantum theory we only ever consider the exponentiated action $e^{iS(\phi)}$, so we'll be okay provided the ambiguity in the WZ term lives in $2\pi\mathbb{Z}$.

   We can ensure this by imposing the condition that k is an integer. Indeed, the ambiguities can be written as integrals of form

   $$\frac{1}{4\pi}\int_{B'} \widetilde{\phi}^*\chi$$

   where B' is a 3-manifold without boundary, by gluing together two maps $\widetilde{\phi}_1, \widetilde{\phi}_2\colon B_{1,2} \to G$ along Σ, where $\partial B_1 = \partial B_2 = \Sigma$, and where $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ are two extensions of ϕ. So we're computing periods of χ along classes in $H_3(G;\mathbb{Z})$. One can check that the relevant period lies in $2\pi\mathbb{Z}$ provided k is an integer. This is a *quantisation condition* for the WZW model.

3. If we compute the classical equations of motion in the WZW model, we find

   $$\overline{\partial}(\phi^{-1}\partial\phi) = 0 \text{ or equivalently } \partial(\phi^{-1}\overline{\partial}\phi) = 0.$$

   In particular, the theory is classically conformally invariant, and classically independent of the value od the level k.

Where does this all come from? One interpretation is that to get these natural equations of motion we'd *like* a 2-form β whose derivative looks like χ, then we could just add to the density a term like $\phi^*\beta$. However χ is not globally exact, so we need to choose an extension of ϕ to make the term we want make sense globally, not just locally. If we try to glue terms like $\phi^*\beta$ together locally we'll find an eventual discrepancy, which we might hope would not make a difference to physically meaningful quantities, but in order for this to be the case we must impose the quantisation condition on k. This should look familiar to people who've studied Dirac's work on magnetic monopoles in the 20s: he made a very similar argument concerning coupling a particle to the electromagnetic field on $\mathbb{R}^3 \setminus \{0\}$ described by a magnetic monopole.

Now, let's investigate what happens if we consider a 2-manifold Σ *with boundary*, for instance the case of an annulus that we're interested in? Given a function $\phi\colon \Sigma \to G$, we can embed our surface Σ in a compact surface Σ' as the complement of a set $D_1, \ldots D_n$ of disjoint discs, and extend ϕ to $\phi'\colon \Sigma' \to G$. The WZ term for g' is now well-defined (up to the issues explained above), so this gives a candidate for the path integral on Σ. However, again we need to investigate how the physics depends on the choice of extension g'! We expect the path integral to depend on a choice of boundary value of the field, which in this case is an element of $(LG)^n$ where LG is the *loop group* of G.

So let ϕ_1 and ϕ_2 be two extensions of ϕ to Σ'. We can set things up so that $\phi_1 = \phi_2 \cdot h$ where $h\colon \Sigma' \to G$ is supported only on the discs $D_1, \ldots D_n$. We can also use Stokes' theorem so see how the WZ term behaves under pointwise multiplication on the target. One finds

$$S_{\mathrm{WZ}}(\phi_1) = S_{\mathrm{WZ}}(\phi_2) + S_{\mathrm{WZ}}(h) + \int_{\Sigma'} \langle \phi_2^{-1} d\phi_2, h^{-1} dh \rangle.$$

The discrepancy can thus be viewed as an integral over a disjoint union of n copies of S^2. We need to deal with the fact that the complex exponential of this discrepancy term may not vanish.

Instead, we define a line bundle over $(LG)^n$ to be the set of equivalence classes in $\mathrm{Maps}(D_1 \sqcup \cdots \sqcup D_n, G) \times \mathbb{C}$ under the equivalence relation

$$(\psi, z) \sim (\psi h, z e^{\frac{i}{12\pi}(S_{\mathrm{WZ}}(h) + \int_{D_1 \sqcup \cdots \sqcup D_n} \langle \psi^{-1} d\psi, h^{-1} dh \rangle)})$$

where h restricts to 1 on the boundary, which lives over $(LG)^n$ by sending a pair (ψ, z) to $\psi|_\partial$. This has been set up so that the exponentiated path integral for the WZW action gives a well-defined section of this line bundle.

It turns out that the total space of this line bundle actually yields admits a natural group structure centrally extending the loop group. On the level of (complexified) Lie algebras this is given by the affine Kac-Moody Lie algebra, defined as follows.

Definition 2.4. The *affine Kac-Moody algebra* associated to a complex semisimple Lie algebra $\mathfrak{g}$ with nondegenerate invariant pairing $\langle\, , \rangle$ is the Lie algebra with underlying vector space

$$\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}((t))) \oplus c\,\mathbb{C},$$

where c is central, and where the brackets of other elements are given by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + c\langle X, Y \rangle \operatorname{res}(f dg).$$

This algebra acts on the Hilbert space of the WZW model, quantising the natural action of the loop group in the classical theory.

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