Topological Quantum Field Theory

And why so many mathematicians are trying to learn QFT

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Introduction and Motivation

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I'm not going to assume you know anything about topology or QFT, and I'll include lots of pictures.

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However turning the arguments into mathematical proofs is hard precisely because there is no mathematical definition of a QFT.

Simplified Models

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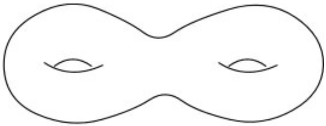
I'll explain the idea in a geometrical way, so I can explain as much as possible through **pictures**.

A Picture of QFT

Start with some geometry. We start with a manifold M (a curved space, whatever dimension we like) which we think of as **spacetime**.



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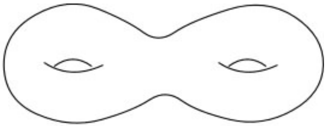
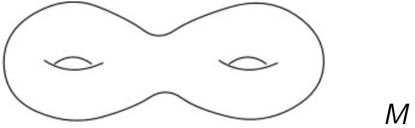


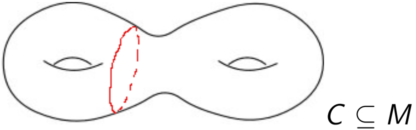
M

We won't assume that M is Minkowski space or anything like that. This picture is popular with string theorists, where M might be the worldsheet of a string (but I'm not making any claims about the physical relevance of string theory).

A Picture of QFT



Whatever **space** is at some instant of time, it should be a **slice** C through spacetime one dimension lower (e.g. M might have a Lorentzian metric, and C might be a Cauchy surface).



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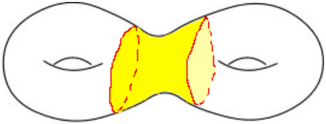
Given two instants in time $t_1 < t_2$, there should be a unitary **time evolution map** between the state spaces. This will depend on the geometry of spacetime in between the two slices.



$$\text{ev}\colon \mathcal{H}(C_1) \to \mathcal{H}(C_2)$$

Topological QFT

Now, we make our simplifying assumption: impose that the Hilbert spaces and time evolution maps only depend on the **topology** of spacetime. That is, only on the shape up to deformations, not on distances, angles, curvature etc.

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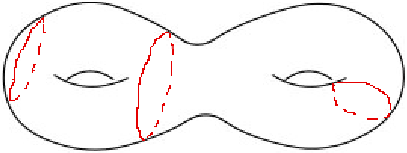
It's easiest to illustrate this by pictures:

Topological QFT



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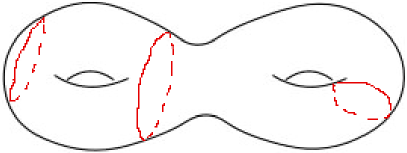

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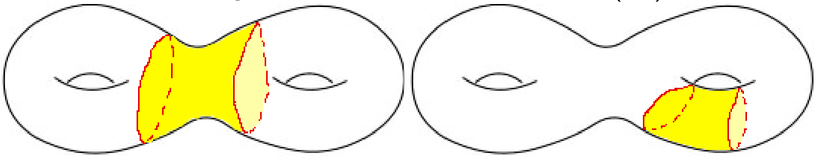
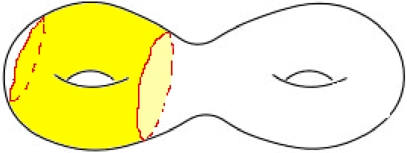


These are the same...



But **this** is different!

Axioms for Topological QFT

Now, let's make this a little more precise (though I'll gloss over many details to keep things simple). I'll describe an n-dimensional TFT, but I'll illustrate everything with 2d pictures.

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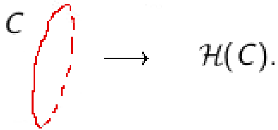
- To every $n-1$-dimensional manifold C we assign a Hilbert space $\mathcal{H}(C)$.



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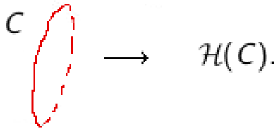
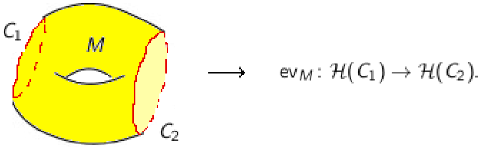


- To a disjoint union $C_1 \sqcup C_2$ we assign the tensor product $\mathcal{H}(C_1) \otimes \mathcal{H}(C_2)$. In particular this means the empty manifold is assigned just $\mathbb{C}$.

Axioms for Topological QFT

- To every n-dimensional manifold M with boundary $C_1 \sqcup C_2$ (we call these things **cobordisms**) we assign a unitary map $\mathrm{ev}_M \colon \mathcal{H}(C_1) \to \mathcal{H}(C_2)$.



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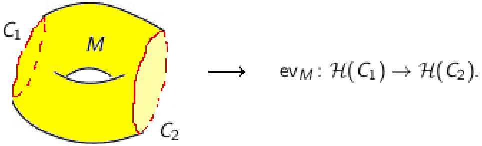

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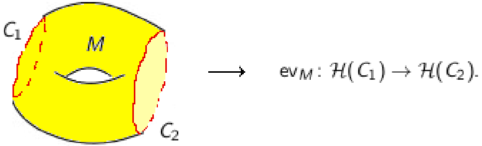

- Cylinders $C \times [0,1]$ are assigned the identity map. (This says the Hamiltonian is trivial!)
- We can **glue** cobordisms together, and the resulting evolution map is just the composite.

An Application to Knots

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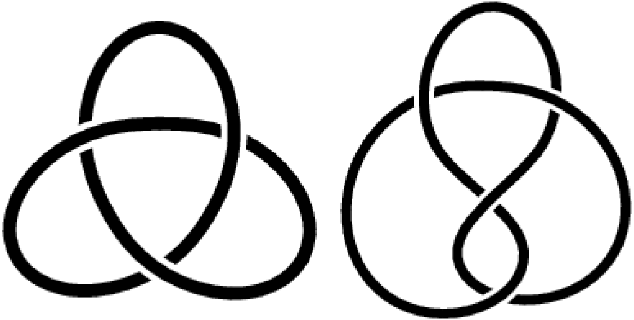
To finish, I'll describe an application due to Witten ('89). He used an example of a TQFT to compute invariants for **knots**. These are numerical ways of distinguishing between different kinds of knots (a hard problem in general).



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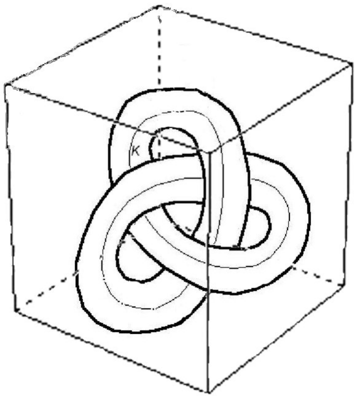
Now, delete a tubular thickening $t(K)$ of K from S^3. The result is a 3d manifold $S^3 - t(K)$ which is **different** for different knots.



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$$\mathrm{ev}_{S^3 - t(K)} \colon \mathcal{H}(T^2) \to \mathbb{C},$$

I.e. a **number** for every state, which is different for different knots.

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One can use this to cook up knot invariants by choosing states in $\mathcal{H}(T^2)$. What's more, one can compute these using path integral methods! So calculations in QFT compute interesting invariants in knot theory.

Thanks for Listening!