The Non-abelian Hodge Correspondence for Non-Compact Curves

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May 8, 2011

1 Setup

In this talk I will describe the non-abelian Hodge theory of a non-compact curve. This was worked out by Simpson in the paper *Harmonic Bundles on Non-Compact Curves*, and as such almost everything I say here can be found in that paper in more detail. Let X be a smooth non-compact curve, with smooth completion $j\colon X \hookrightarrow \overline{X}$, and $S = \overline{X} \setminus X$ a finite set of punctures. For simplicity, I'll assume we have only a single puncture $S = \{s\}$. To produce a non-abelian Hodge correspondence in this setting, we'll need to consider *filtered* analogues of flat and Higgs bundles, and impose tameness conditions on the behaviour of the Higgs field or connection with respect to this filtration.

One way to think about objects like Higgs and flat bundles on non-compact varieties X is as *singular* objects living on a smooth completion $\overline{X}$, with singularities supported along the divisor $\overline{X} \setminus X$. The idea of *tameness* or *regularity* is to control how badly behaved these singularities are allowed to be.

In what follows, let U be a small contractible neighbourhood of the puncture s in $\overline{X}$. Let z be a local coordinate on U vanishing at s.

Definition 1.1. A *filtered vector bundle* on X is a locally free sheaf E on X, equipped with a decreasing filtration

$$\bigcup_{\alpha \in \mathbb{R}} E_\alpha$$

of j_*E by coherent subsheaves satisfying

- $E_\alpha = \bigcap_{\beta < \alpha} E_\beta$
- $E_{\alpha+1} = zE_\alpha$ (so in particular the filtration is determined by $\alpha \in [0, 1)$.)

One could view this as a collection of extensions of the vector bundle E on X across the puncture s that form a decreasing left continuous filtration.

In this vein, a filtered Higgs or flat bundle is just a filtered vector bundle on X with a Higgs field θ or flat connection ∇. For the moment we do not impose any conditions relating this additional data to the filtration. If we allowed multiple punctures the definitions would be extremely similar, we would merely ask for a filtration on every $j_{s_i,*}E$, one for each puncture.

Remark 1.2. I could define the notion of a *filtered local system* as the Betti analogue of a filtered flat bundle: wherever I describe correspondences between flat and Higgs bundles in this setting there is always also a local systems analogue. For brevity, I will omit these details.

How can we use a filtration to control the local behaviour (e.g. "growth rate") of something like a connection or Higgs field near the puncture? To see this, we will consider the classical notion of *regular singularities*.

2 Regularity and Tameness

First, we consider the notion of *tameness* for a harmonic bundle on X as our main way of controlling growth rates of sections near the puncture. Tameness means that the Higgs field θ associated to a harmonic bundle has eigenvalues that have *moderate growth* near the puncture:

$$|\lambda| \leq \frac{c}{r} \quad \forall \varepsilon > 0$$

where $r = |z|$ denotes the distance from the puncture in our standard local coordinate system: as when we described the notion of moderate growth above, this should hold in angular sectors about the puncture, ensuring that the λ are single-valued functions. The reason the tameness/regularity hypothesis is needed is, in the course of the proof of this fact we need to establish analytic estimates on the size of these eigenvalues. I'll say a little more about this shortly.

This eigenvalue definition of tameness is equivalent to the following notion: the data of a harmonic bundle includes a $\pi_1(X)$ equivariant map

$$F \colon \widetilde{X} \to GL(n, \mathbb{C})/U(n).$$

We say the harmonic bundle is *tame* if whenever we choose locally a *ray* ρ extending out from the puncture, we can lift $F|_\rho$ to a map $\rho \to GL(n, \mathbb{C})$ whose image grows in norm at most polynomially in $1/r$, where r is the distance out from the puncture along ρ.

We'll need to restrict to the tame case to prove a non-abelian Hodge theorem over X: in the course of the proof it will be necessary to establish analytic estimates on things like the curvature of our bundles in certain metrics so that the natural functors between harmonic bundles and filtered flat or Higgs bundles are well-defined. Moderate growth is needed to establish these estimates. So with this in mind we can work out what restrictions on flat and Higgs bundles we need to introduce to correspond to this moderate growth condition. This condition will be called *regularity.*

Regular singularities arise as a notion in the study of systems of linear ODEs. Let K denote the field of meromorphic functions holomorphic away from zero, i.e. K is the field of fractions of the stalk $\mathcal{O} = (\mathcal{O}_\mathbb{C})_0$. More concretely, $K \cong \mathbb{C}\{\{z\}\}[z^{-1}]$ where $\mathbb{C}\{\{z\}\}$ denotes the ring of *convergent* power series at $z = 0$. Consider a differential equation

$$P = \sum_{i=o}^{n} a_i(x) \left(\frac{d}{dx}\right)^{n-i}$$

for $a_i \in K$, $a_0 \neq 0$. This is equivalent to a system of linear first order equations, which we can express in the form

$$\frac{d}{dx} u_i(x) = \sum_{j=0}^{n} a_{ij}(x) u_j(x),$$

for $i = 1, \ldots, n$, or equivalently

$$\frac{d}{dx} u(x) = A(x) u(x)$$

for $A \in \mathrm{Mat}_n(K)$. Two such systems $\frac{d}{dx} u(x) = A_i(x) u(x)$, $i = 1, 2$ are called *equivalent* if there exists a matrix $T \in GL_n(K)$ such that

$$A_1 = T A_2 T^{-1} - T \frac{d}{dx} T^{-1}$$

which is simply a change of variable condition: the result of setting $v(x) = T u(x)$.

Classically, a system of ODEs was called *regular* or Fuchsian if its solutions have a property called *moderate growth.* To be precise,

Definition 2.1. Consider a *multivalued solution* $u \in \widetilde{\mathcal{O}}$ of the system $\frac{d}{dx} u(x) = A(x) u(x)$. The solution u is said to have *moderate growth* if for any sector of the form

$$S = \{z = (r, \theta) \,:\, 0 < r < \varepsilon,\, \theta_0 < \theta < \theta_1\},$$

there exists a constant $c > 0$ and an integer $j \in \mathbb{N}$ such that

$$|u(z)| < \frac{c}{|z|^j}$$

for all $z \in S$. If the solutions of the system have moderate growth, we say the system is *regular*, or has regular singularity at 0.

The following is a classical result on regularity.

Theorem 2.2. The following are equivalent:

1. The system
$$\frac{d}{dx}u(x) = A(x)u(x) \tag{1}$$
is regular.
2. The system (1) is equivalent to one of the form
$$\frac{d}{dx}u(x) = \frac{B(x)}{x}u(x)$$
where $B(z)$ is a matrix with holomorphic coefficients.
3. The system (1) is equivalent to one of the form
$$\frac{d}{dx}u(x) = \frac{C}{x}u(x)$$
where C is a matrix with constant coefficients.

Theorem 2.3 (Fuchs)**.** The system $\frac{d}{dx}u(x) = A(x)u(x)$ is equivalent to a single equation $\sum_{i=0}^{n} a_i(x)\left(\frac{d}{dx}\right)^{n-i} u = Pu = 0$. It is regular if and only if $\frac{a_i}{a_0}$ has a pole at 0 of order at most i, for $i = 1, \ldots, n$.

More immediately applicable to our purposes is the following reformulation. A system of linear ODEs of this form is equivalent to a vector bundle over $\mathbb{C}^\times$ equipped with a *meromorphic connection*:

Definition 2.4. A *meromorphic connection* on $\mathbb{C}$ at 0 is a finite dimensional vector space M over K equipped with a $\mathbb{C}$-linear map $\nabla\colon M \to M$ such that
$$\nabla(fm) = \frac{df}{dx}m + f\nabla(m) \quad \forall\, f \in K,\ \forall\, m \in M.$$
Two such meromorphic connections (M_1, ∇_1) and (M_2, ∇_2) are *isomorphic* if there exists an isomorphism $\phi\colon M_1 \to M_2$ such that $\phi \circ \nabla_1 = \nabla_2 \circ \phi$.

In terms of meromorphic connections, this means we can make the following definition:

Definition 2.5. Let (M, ∇) be a meromorphic connection. We say the connection is *regular* (at 0) if there exists a basis $e_1, \ldots, e_n$ of M over K such that
$$\nabla e_i = -\sum_j \frac{b_{ij}(z)}{z} e_j \quad b_{ij} \in \mathcal{O}.$$
Equivalently, there exists a finitely generated submodule L of M such that $z\nabla L \subseteq L$, generating M over K.

In view of the previous two theorems, we can see that this is equivalent to the regularity of the associated system of linear differential equations. Indeed, the meromorphic connection corresponds to the system
$$\frac{du_i}{dx} = \sum_j \frac{b_{ij}(z)}{z} u_j$$
which, by 2.2, is regular precisely when we can choose $b_{ij} \in \mathcal{O}$, i.e. when there exists a basis of the above form for the associated meromorphic connection.

This should motivate the following definition of regularity as imposing moderate growth on flat sections:

Definition 2.6. A connection ∇ on the filtered vector bundle $E = \bigcup E_\alpha$ is called *regular* if on the filtered pieces it maps

$$\nabla\colon E_\alpha \to E_\alpha \otimes \Omega^1_{\overline{X}}(\log s)$$

where $\Omega^1_{\overline{X}}(\log s)$ is the sheaf of differentials generated by $\frac{dz}{z}$, for z as usual a local coordinate vanishing at the puncture s: the sheaf of *logarithmic differentials*. This means in particular that the action of $z\frac{\partial}{\partial z}$ preserves the filtration.

Similarly, a Higgs field θ on E is called *regular* if it maps

$$\theta\colon E_\alpha \to E_\alpha \otimes \Omega^1_{\overline{X}}(\log s).$$

A regular filtered flat bundle is in particular a holonomic D-module on $\overline{X}$ with a regular singularity at the point s. Locally, this is the same as a meromorphic connection with regular singularity, as described above (or rather, the algebraic analogue of the analytic definition given above).

These regular objects correspond to *tame* harmonic bundles on X, and thus we have a non-abelian Hodge theorem in this setting.

Theorem 2.7 (Non-abelian Hodge correspondence for a non-compact curve)**.** There is a natural equivalence of categories between stable regular filtered Higgs bundles of degree zero and stable regular filtered flat bundles of degree zero on the curve X. By the *degree* of a filtered vector bundle, we mean its *algebraic degree*

$$\deg(E, E_\alpha) = \deg(E_0) + \sum_{\alpha\in[0,1)} \mathrm{rk}(\mathrm{gr}_\alpha(E_s)).$$

The correspondence is essentially the same as in the compact case, and the filtrations on the two sides are the same. One proves that both sides are equivalent to irreducible tame harmonic bundles on the punctured curve. What is interesting and new in this setting is the correspondence also neatly relates the growth behaviour of the Higgs field and the connection near the puncture s, in a way we will attempt to make precise. Before we do this, let me say a few words about a tame harmonic bundle gives a filtration on the underlying vector bundle, with respect to which the usual constructions of θ and ∇ will be regular. If E is a holomorphic vector bundle with a metric K, following Simpson we define $\Xi(E)$ to be the filtered vector bundle where the germs of $\Xi(E)_\alpha$ at s are those sections of E in a punctured neighbourhood of s satisfying the growth condition

$$|e|_K \leq Cr^{\alpha-\varepsilon} \quad \text{for all } \varepsilon > 0.$$

This is where tameness comes in: we use the analytic estimates I mentioned earlier to ensure that this functor is well-defined: i.e. that the $\Xi(E)_s$ are *coherent* subsheaves of j_*E.

3 Residues

So far all we have seen is that in a suitably restricted setting – where tameness is imposed – a non-abelian Hodge correspondence still holds. However, we can actually say something stronger. By defining a *residue* that captures the limiting behaviour of the objects near the puncture s, we can actually show that the local behaviour of the Higgs, flat and harmonic bundles near s are related in a rigid way. Let me give a definition describing what data we have at a singularity, then unpack it to see what it actually implies.

Definition 3.1. Let (E, E_α, θ) be a regular filtered Higgs bundle. The *residue* of E at the puncture s is the pair $(\mathrm{res}(E), \mathrm{res}(\theta))$ consisting of the graded vector space

$$\mathrm{res}(E) = \bigoplus_{\alpha\in[0,1)} \mathrm{gr}_\alpha(\overline{E})$$

the associated graded of the filtered extension $\overline{E}$ of E to $\overline{X}$ given by pushing forward, and the endomorphism $\mathrm{res}(\theta) = z\theta(\frac{\partial}{\partial z})$. Here we are using the fact that E is regular to produce an endomorphism of the graded vector space.

Similarly, let (E, E_α, ∇) be a regular filtered flat bundle. We can similarly define its residue to be the same graded vector space, but with endomorphism $\mathrm{res}(\nabla)$ given by the action of $z\nabla(\frac{\partial}{\partial z})$ on the associated graded. More explicitly, this is the residue evaluation given by taking

$$\mathrm{gr}(\nabla)\colon\ \mathrm{gr}(E_\alpha) \to \mathbb{C}\langle \frac{dz}{z} \rangle$$

and evaluating $dz/z \mapsto 1$.

In both cases we are essentially just killing the $\frac{dz}{z}$ part of the stalk of $E \otimes \Omega(\log z)$.

Let me try to explain how to quantify this residue data and how it relates to the local behaviour of the Higgs field or connection by means of a a simple example.

Example 3.2. Suppose E is a *line bundle* on X with filtration E_α. Then one can find a point $j \in [0, 1)$ such that for all $\varepsilon > 0$ $E_{j+\varepsilon} \neq E_j$, i.e. the filtration *jumps* at j. Since E is a line bundle this point is unique. So $\mathrm{res}(E) = E_j$. If we have a regular Higgs field or flat connection on E, its residue is an endomorphism of a one-dimensional vector space, i.e. a number. Thus our residue data consists of two numbers:

1. a real number $j \in [0, 1)$: the *jump*.
2. a complex number $\lambda \in \mathbb{C}$: the *eigenvalue*.

What do these numbers say about the Higgs field or connection? First of all, recall where the filtration E_α came from, from the point of view of the harmonic bundle. A germ of E in a punctured neighbourhood of s being in E_α but not $E_{\alpha+\varepsilon}$ meant that in the harmonic metric, it grew at least as fast as r^α, but not as fast as $r^{\alpha+\varepsilon}$ for all $\varepsilon > 0$. So the jump of our bundle should tell us about the growth rate of local sections of E in the harmonic metric.

In a more complicated situation: a rank n vector bundle, we can describe the data we have as follows: the jumps in the filtration – i.e. the points in $[0, 1)$ at which $\mathrm{gr}(E)$ is supported – give a partition of $\mathrm{gr}(E)$ into vector spaces E_j of sizes summing to n. This decomposition reflects the possible polynomial growth rates of the sections of E in the harmonic metric. Each of these pieces admits a further decomposition into generalised eigenspaces of the endomorphism $\mathrm{res}(\theta)$ or $\mathrm{res}(\nabla)$, so the pairs (j, λ) describe a labelled partition $P_{j,\lambda}$ of n. However, there is still more information. On each such piece, the endomorphism acts like a diagonal matrix plus a nilpotent, e.g.

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

This nilpotent part contains further information, which can be described in a canonical way by a *weight filtration*. But before we describe this, let me describe how the coarser part – the jumps and eigenvalues – of the residue relate under the non-abelian Hodge correspondence: essentially they are preserved, up to a permutation which one can explicitly compute. This is what I meant earlier when I mentioned "rigidity" of the non-abelian Hodge correspondence.

Example 3.3. Lets consider the example of the line bundle once again: so there is a single block $P_{j,\lambda}$ labelled by a real and a complex number in each of the Higgs and flat settings. The correspondence is:

	Higgs	Flat
	j	$j - 2b$
λ	$a + ib$	$j + 2bi$

This is a reasonably simple computation on the growth rate of holomorphic sections of the line bundle in the standard metric, carried out in section 5 of Simpson's paper.

I'll say a few words about the weight filtration on the nilpotent pieces, and their preservation under the correspondence.

Definition 3.4. On such a $P_{j,\lambda}$, there exists a unique increasing exhaustive filtration W_k, $k \in \mathbb{Z}$ – the *weight filtration* – such that the endomorphism $\operatorname{res}(\theta)$ or $\operatorname{res}(\nabla)$ lowers weights by two, and so that the weights of each Jordan block are arranged symmetrically about the origin.

Example 3.5. The crucial example is the harmonic rank 2 bundle over the punctured disc corresponding to the variation of Hodge structure W given by the standard representation of SL_2. In this case we can choose a basis such that the residue endomorphism $N = \operatorname{res}(\theta)$ has form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Notice this is $\mathbb{C}^\times$ invariant, so does indeed underly a variation of Hodge structure. Here there are two non-trivial pieces of the weight filtration:

$$\begin{aligned} W_{<-1} &= 0 \\ W_{-1} &= \ker(N) = W_0 \\ W_1 &= W. \end{aligned}$$

The reason this is crucial is, by taking symmetric powers of this VHS we can produce nilpotent Jordan blocks of arbitrary size. Then twisting by line bundles and taking direct sums allows us to produce all possible residue data as arising from harmonic bundles.

4 Some Remarks on the Higher-Dimensional Case

How does this generalise to the case where X is a higher-dimensional variety, so we have to consider singularities in the neighbourhood of a divisor, not just an isolated point? Let me begin to discuss what kind of thing one can say, to be elaborated on in subsequent talks. Firstly, there is a natural extension of the concept of regular singularities for D-modules on higher dimensional varieties, generalising the notion we have discussed. A flat bundle on X is *regular* if for every morphism $f : C \to X$ where C is a smooth algebraic curve, the pullback of the flat bundle under f is regular, in the sense we have already considered. This further generalises to all holonomic D-modules by a classification theorem: a composition series is given where the composition factors are simple D-modules, we say the D-module is *regular* is these simple D-modules are minimal extensions of flat bundles on affine open subvarieties.

Simpson proved that on a general Kähler manifold – not necessarily compact – there is an equivalence of categories between harmonic bundles and pure polarized twistor structures of degree zero. In the tame case, one can say something stronger:

Theorem 4.1 (Tame harmonic bundles are equivalent to twistor D-modules)**.** Let $j : X \hookrightarrow \overline{X}$ be a smooth completion of X, with complement $N = \overline{X} \setminus X$ a normal crosssings divisor. Then we know flat connections (E, ∇) on X are equivalent to so-called "variations of pure polarizable degree zero twistor structures" on X –a generalisation of the notion of a variation of Hodge structure. But in fact, if ∇ is regular on N, then the corresponding variation of twistor structure extends to a "twistor D-module" on $\overline{X}$.

Generalising this, Mochizuki proved that on a quasiprojective variety, there is an equivalence between polarizable twistor D-modules of weight zero and semisimple regular holonomic D_X-modules: this is a generalisation of the above theorem, which considered regular holonomic D-modules which restrict to flat connections on the complement of a normal crosssings divisor. Twistor D-modules can be thought of as something like λ-connections, parameterised by $\lambda \in \mathbb{P}^1$, and the functor here is simply restriction to $\lambda = 1$.