Math 121 – Calculus II – Tests for Convergence and Divergence

1. n^{th} **Term Divergence Test**: Given a series $\sum_{n=1}^{\infty} a_n$, if $\lim_{n\to\infty} a_n \neq 0$ then the series always *diverges*.

2. **Geometric Series**: A geometric series is a series of the form

$$\sum_{n=0}^{\infty} ar^n.$$

   Here a is the *first term* of the series and r is the *common ratio*. You can check that a series $\sum_{n=0}^{\infty} a_n$ is geometric by checking whether a_{n+1}/a_n is *constant*. A geometric series *diverges* if $|r| \geq 1$ and *converges* if $|r| < 1$. A convergent geometric series converges to

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

3. **Integral Test**: Suppose you have a function f so that $f(n) = a_n$ for all $n \geq 1$. Suppose that
   - f is continuous for $x \geq 1$,
   - f is decreasing for $x \geq 1$ (for instance f is differentiable and $f'(x) \leq 0$), and
   - f is positive for $x \geq 1$ (i.e. $f(x) \geq 0$).

   Then the series $\sum_{n=1}^{\infty} a_n$ converges/diverges if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ converges/diverges.

4. p**-Series**: Using the integral test we showed that the series

$$\sum_{n=1}^{\infty} n^p$$

   converges if $p < -1$ and *diverges* if $p \geq -1$.

5. **Comparison Test**: Given two series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} A_n,$$

   suppose that $a_n \geq 0$, $A_n \geq 0$ and $a_n \leq A_n$ for all n. Then:
   - If the small series $\sum_{n=1}^{\infty} a_n$ *diverges*, so does the large series $\sum_{n=1}^{\infty} A_n$.
   - If the large series $\sum_{n=1}^{\infty} A_n$ *converges*, so does the small series $\sum_{n=1}^{\infty} a_n$.

6. **Limit Comparison Test**: Given two series

$$\sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n,$$

   suppose that $a_n \geq 0$, $b_n \geq 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k, \text{ not equal to } 0 \text{ or } \infty.$$

   Then $\sum_{n=1}^{\infty} a_n$ converges/diverges if $\sum_{n=1}^{\infty} b_n$ converges/diverges.

7. **Alternating Series Test**: An *alternating series* is a series of the form

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n |a_n| \text{ or } \sum_{n=1}^{\infty} (-1)^{n+1} |a_n|.$$

   Given an alternating series, if

   - $\lim_{n\to\infty} |a_n| = 0$ and
   - $|a_n|$ is decreasing,

   then the alternating series $\sum_{n=1}^{\infty} a_n$ converges.

8. **Ratio Test**: Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ for all n, we compute the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

   - If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges.
   - If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $L = 1$ the ratio test is inconclusive.

9. **Absolute Convergence**: We say a series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$. *Any absolutely convergent series is convergent*. A convergent series that is not absolutely convergent is called *conditionally convergent*.