Math 132H – Homework 4

Due: Wednesday October 14th

You should explain your reasoning carefully using English sentences where appropriate, not only equations. You may use the textbook and your notes, and you're welcome to discuss the problems with one another, with me, and with the TA, but your final answers should be your own and in your own words

Let a_0 and b_0 be two positive real numbers with $a_0 > b_0$. The *arithmetic mean* of a_0 and b_0 is

$$a_1 = \frac{a_0 + b_0}{2},$$

and the *geometric mean* of a_0 and b_0 is

$$b_1 = \sqrt{a_0 b_0}.$$

1. For any values of a_0 and b_0, show that $a_1 > b_1$. This is a special case of the *arithmetic mean - geometric mean (AMGM) inequality* which says that the same inequality holds for the arithmetic and geometric mean of more than two numbers (you don't need to prove this more general fact). Hint: start with the fact that $(a_0 - b_0)^2 > 0$.

2. Define sequences (a_n) and (b_n) by letting

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \qquad b_n = \sqrt{a_{n-1} b_{n-1}}.$$

   Deduce from the previous part that

$$a_n > b_n$$

   for all n, and hence also that

$$a_n > a_{n+1} \text{ and } b_{n+1} > b_n$$

   for all n.

3. Use the monotone convergence theorem to show that the sequences (a_n) and (b_n) are both convergent to the same value $M(a_0, b_0)$. This $M(a_0, b_0)$ is called the *arithmetic-geometric mean* of a_0 and b_0.

   The arithmetic-geometric mean is very useful for computers to calculate with exponential and trigonometric functions. It turns out (although we won't prove) that

$$\frac{1}{M(a_0, b_0)} = \frac{2}{\pi} \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{a_0^2 \cos^2(x) + b_0^2 \sin^2(x)}}.$$

   Computers use the sequences (a_n) and (b_n) to estimate the right-hand side, which can be used to derive quick algorithms for computing exponential and trigonometric functions to many decimal places.

4. If $a_0 = 1$ and $b_0 = \sqrt{2}$, use the first few values of a_n and b_n to estimate $M(1, \sqrt{2})$ to 3 decimal places. The number $G = \frac{1}{M(1,\sqrt{2})}$ is called *Gauss's constant*: Gauss used it to compute the arc-length of certain curves (such as the *leminscate*: a figure 8 shaped curve described by a quartic function of x and y).