Math 132H – Homework 5

Due: Wednesday October 28th

You should explain your reasoning carefully using English sentences where appropriate, not only equations. You may use the textbook and your notes, and you're welcome to discuss the problems with one another, with me, and with the TA, but your final answers should be your own and in your own words

In this homework we'll learn about a new convergence test called *Cauchy's Condensation Test*. Suppose $\sum_{n=1}^{\infty} a_n$ is a series where the terms a_n are ≥ 0 and decreasing, so $a_{n+1} \leq a_n$ for all n. The Cauchy condensation test says that

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \sum_{n=1}^{\infty} 2^n a_{2^n} \text{ converges.}$$

This test generalizes the idea from Oresme's proof of the divergence of the harmonic series where you group terms together in groups of size $1, 2, 4, 8$ etc.

1. By grouping the terms a_n together into groups of size $1, 2, 4, \ldots, 2^N$, argue that

$$\sum_{n=1}^{2^{N+1}-1} a_n \leq \sum_{n=0}^{N} 2^n a_{2^n}.$$

2. By taking the limit $N \to \infty$, argue that if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges then so does $\sum_{n=1}^{\infty} a_n$. (Hint: apply the monotone convergence theorem to the sequence of partial sums of the series $\sum_{n=1}^{\infty} a_n$.)

   This is half of the proof that the Cauchy condensation test works. A similar argument can be used to show that if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ diverges then so does $\sum_{n=1}^{\infty} a_n$. You don't have to show this.

3. Consider the series

$$\sum_{n=2}^{\infty} n^p \log_2(n)^q$$

   where p and q are ≤ 0. Use the ordinary comparison test to show that the series converges if $p < -1$. Here $\log_2$ means the logarithm with base 2.

4. Use the Cauchy condensation test to show that if $p = -1$ then this series converges if $q < -1$ and diverges if $q \geq -1$.

5. Use the Cauchy condensation test twice to show that

$$\sum_{n=3}^{\infty} \frac{1}{n \log_2(n) \log_2(\log_2(n))}$$

   diverges.