# Math 132H – Homework 6

**Due: Wednesday November 11th**

You should explain your reasoning carefully using English sentences where appropriate, not only equations. You may use the textbook and your notes, and you're welcome to discuss the problems with one another, with me, and with the TA, but your final answers should be your own and in your own words

1. Consider the function $K(x)$ defined as a power series:

$$K(x) = \frac{\pi}{2}\left(1 + \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 x^n\right),$$

where $(2n)!! = 2n(2n-2)(2n-4)(2n-6)\cdots 2$ and $(2n-1)!! = (2n-1)(2n-3)(2n-5)\cdots 1$ denote the "double factorial" (like the factorial, but you only include every other number).

This function is called an *elliptic integral*: it computes the perimeter of an ellipse, or the period of oscillation of a pendulum (where we set $x = \sin^2(\theta/2)$ for $\theta$ the amplitude of the oscillation).

   (a) Compute the radius of convergence of the power series defining $K(x)$.

   (b) Compute $L = \lim_{n\to\infty} \frac{(2n-1)!!}{(2n)!!}$ (Hint: try computing $\ln(L)$.)

   (c) Test whether the power series converges at $x = -1$.

2. The *Airy function* $\mathrm{Ai}(x)$ is defined by the following power series:

$$\mathrm{Ai}(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{(2\cdot 3)\cdot(5\cdot 6)\cdot(8\cdot 9)\cdots((3n-1)\cdot 3n)} x^{3n}.$$

   (a) Show that the power series defining $\mathrm{Ai}(x)$ converges for all $x$.

   (b) Compute the second derivative $\mathrm{Ai}''(x)$.

   (c) Check that $y = \mathrm{Ai}(x)$ is a solution to the equation

$$y'' - xy = 0.$$

   This is called *Airy's equation*: it occurs when studying diffraction of light through a circular aperture, and when solving Schrödinger's equation in quantum mechanics with certain simple potentials.