Math 211 – Multivariate Calculus – Homework 2

Due: Friday September 16th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two points in $\mathbb{R}^3$, write an equation describing the set of points which are the same distance away from $\boldsymbol{u}$ and $\boldsymbol{v}$. Draw a picture of this set in the case where $\boldsymbol{u} = \boldsymbol{i}$ and $\boldsymbol{v} = \boldsymbol{j}$.

2. Define the *scalar triple product* of three vectors to be $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$.

   (a) Show that the triple product is unchanged by *cyclically permuting* the three vectors. In other words, check that
   $$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}).$$

   (b) Check that the triple product $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$ is equal to zero if and only if the three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ all lie within a plane.

   (c) Use the triple product to check that the vectors $\boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$, $3\boldsymbol{i} - \boldsymbol{j}$ and $5\boldsymbol{i} + 9\boldsymbol{j} - 4\boldsymbol{k}$ all lie within a plane.

3. Suppose that $\boldsymbol{u}$ is a non-zero vector in $\mathbb{R}^3$. Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two other vectors.

   (a) Suppose that $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{w}$. Does it follow that $\boldsymbol{v} = \boldsymbol{w}$? Why or why not?

   (b) Suppose that $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{w}$. Does it follow that $\boldsymbol{v} = \boldsymbol{w}$? Why or why not?

   (c) Suppose that $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{w}$ *and* $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{u} \times \boldsymbol{w}$. Does it follow that $\boldsymbol{v} = \boldsymbol{w}$? Why or why not?·

4. Consider a triangle in $\mathbb{R}^3$ with vertices A, B and C. Write $\boldsymbol{u}$ for the vector from A to B, $\boldsymbol{v}$ for the vector from B to C and $\boldsymbol{w}$ for the vector from C to A.

   (a) Show that
   $$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{w} \times \boldsymbol{u}.$$

   (b) Deduce from this the law of sines, which says that if a, b, c are the side lengths of a triangle and α, β, γ are the angles (with angle α appearing opposite side a etc), then
   $$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}.$$

   (c) Now, suppose that $\boldsymbol{p}, \boldsymbol{q}$ and $\boldsymbol{r}$ are *any* three vectors in $\mathbb{R}^3$ with the property that $\boldsymbol{p} \times \boldsymbol{q} \neq \boldsymbol{0}$, and with the property that
   $$\boldsymbol{p} \times \boldsymbol{q} = \boldsymbol{q} \times \boldsymbol{r} = \boldsymbol{r} \times \boldsymbol{p}.$$
   Show that $\boldsymbol{p} + \boldsymbol{q} + \boldsymbol{r} = \boldsymbol{0}$.