Math 211 – Multivariate Calculus – Homework 4

Due: Friday September 30th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Recall that in class we derived the equations

$$\boldsymbol{T}' = \kappa \boldsymbol{N}, \quad \boldsymbol{B}' = -\tau \boldsymbol{N},$$

where in this problem I am using $'$ to indicate differentiation with respect to s.

   (a) Show (using the fact that $\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T}$) that the following formula also holds:

$$\boldsymbol{N}' = -\kappa \boldsymbol{T} + \tau \boldsymbol{B}$$

   Taken together, these three equations are known as the *Frenet–Serret equations*.

   (b) Use the equation $\boldsymbol{B}' = -\tau \boldsymbol{N}$ to deduce that $\tau = -\boldsymbol{B}' \cdot \boldsymbol{N}$, and hence derive the following formula

$$\tau = \frac{(\boldsymbol{r}' \times \boldsymbol{r}'') \cdot \boldsymbol{r}'''}{\|\boldsymbol{r}' \times \boldsymbol{r}''\|^2}.$$

2. If f is a function from $\mathbb{R}$ to $\mathbb{R}$, we can define its graph as a *plane curve* using the parametric formula $\boldsymbol{r}(t) = (t, f(t), 0)$.

   (a) If f is twice differentiable, show that the curvature of the plane curve is given by the formula

$$\kappa = \frac{|f''(t)|}{(1 + (f'(t))^2)^{3/2}}.$$

   You may use Theorem 10 in chapter 13.3 of the textbook if you like.

   (b) Find the point on the plane curve $\boldsymbol{r}(t) = (t, e^t, 0)$ where the curvature is maximised. What happens to the curvature as $t \to \infty$?

   (c) Find the point on the plane curve $\boldsymbol{r}(t) = (t, \ln(t), 0)$ where the curvature is maximised. What happens to the curvature as $t \to \infty$?

3. Consider the curve defined by $\boldsymbol{r}(t) = (3t - t^3, 3t^2, 3t + t^3)$.

   (a) Find the arc length of the curve from $t = 0$ to $t = b$.

   (b) Find an equation for the *osculating plane* to the curve at $t = 1$.

   (c) Find an equation for the *normal plane* to the curve at $t = 1$. That is, the plane containing $\boldsymbol{N}$ and $\boldsymbol{B}$.

   (d) Find an equation for the *rectifying plane* to the curve at $t = 1$. That is, the plane containing $\boldsymbol{T}$ and $\boldsymbol{B}$.

4. Find κ and τ for the curve $\boldsymbol{r}(t) = (t - \sin(t), 1 - \cos(t), 4\sin(t/2))$