Math 211 – Multivariate Calculus – Homework 5

Due: Friday October 14th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Let $\boldsymbol{F}\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the function defined by

$$\boldsymbol{F}(x, y) = (f(x, y), g(x, y), h(x, y)) = (2x^2y - x^4, e^{xy} - y\sin(x), x^2\cos(x)).$$

   Find all second partial derivatives of $\boldsymbol{F}$. You can compute these partial derivatives component by component, so for instance

$$\boldsymbol{F}_x = (f_x, g_x, h_x),$$

   and so on.

2. (a) Show that a differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ decreases most rapidly at a point $\boldsymbol{x}$ in the direction

$$\boldsymbol{v} = -\nabla f(\boldsymbol{x}).$$

   (Hint: the rate of decrease is given by the directional derivative $\mathrm{d}_{\boldsymbol{v}} f(\boldsymbol{x})$. So figure out for which unit vector $\boldsymbol{v}$ this directional derivative is minimal.)

   (b) Find the direction in which the function $f(x, y) = x^4y - x^2y^3$ decreases the most rapidly at the point $(2, -3)$.

3. Find an equation for the tangent plane to the surface with equation

$$z = \ln(x - 3y)$$

   at the point $(x, y, z) = (4, 1, 0)$.

4. A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is called *homogeneous of degree* n if

$$f(tx, ty) = t^n f(x, y)$$

   for all values of t. Suppose f has continuous second-order partial derivatives.

   (a) Check that $f(x, y) = x^2y + 2xy^2 + 5y^3$ is homogeneous of degree 3.

   (b) If f is homogeneous of degree n show that

$$f_x(tx, ty) = t^{n-1} f_x(x, y).$$

   (c) By considering $\frac{\partial}{\partial t} f(tx, ty)$, show that if f is homogeneous of degree n then

$$x f_x(x, y) + y f_y(x, y) = n f(x, y).$$