Math 211 – Multivariate Calculus – Homework 8

Due: Friday November 11th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Express the integral $\iiint_R f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ in six different ways, where R is the solid bounded by:
   (a) $y = x^2, z = 0, y + 3z = 4$.
   (b) $x = 2, y = 2, z = 0, x + y - 2z = 2$.
2. Compute $\iint_R (2 + \cos^6(x)\sin^3(y) - 6x^3y^2e^{x^4})\mathrm{d}x\mathrm{d}y$, where
$$R = \{(x,y) : |x| + |y| \le 1\}.$$
   You should consider using *symmetry* to simplify the calculation.
3. Find the mass and center of mass of the following solids
   (a) The region above the xy plane and below the paraboloid with equation $z + x^2 + y^2 = 1$, with constant density $\rho = 3$.
   (b) The tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$ with density function $\rho(x,y,z) = y$.
4. *Gaussian integral:* In this exercise we will prove the formula
$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$
   (a) We define an improper integral over the plane $\mathbb{R}^2$ as a limit:
$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y = \lim_{a\to\infty}\iint_{D_a} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y,$$
   where D_a is the disk around 0 of radius a. Use this definition to prove that $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y = \pi$.
   (b) We can equivalently compute the improper integral as a limit over regions of a different shape:
$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y = \lim_{a\to\infty}\iint_{S_a} e^{-x^2-y^2}\mathrm{d}x\mathrm{d}y,$$
   where S_a is the square $[-a,a]\times[-a,a]$. Use the previous part to deduce that
$$\left(\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y\right) = \pi.$$
   (c) Hence deduce that
$$\int_{-\infty}^{\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}.$$