Math 250 – Number Theory – Homework 1

Due: Friday February 10th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Use Euclid's algorithm to find the greatest common divisor of the following pairs of integers a, b. Then solve Bézout's equation by finding integers x, y so that $\gcd(a, b) = ax + by$.
   (a) $a = 74, b = 383$.
   (b) $a = 687, b = 24$.
   (c) $a = 7544, b = 115$.
2. (a) Prove the following converse to Bézout's theorem. If a and b are integers, suppose one can find integers x, y solving the equation
   $$1 = ax + by.$$
   Then a and b are coprime.
   (b) Hence prove that if a, b, c are integers such that a and b are coprime, and a and c are coprime, then a and bc are coprime.
3. (a) Let a and b be coprime integers with $a > 1$ and $b > 1$. Prove that every integer c can be written in the form $ax + by$ for some integers x and y in infinitely many ways.
   (b) Show that the equation $ax + by = 1$ has a solution in which $|x| < b$ and $|y| < a$.
   (c) Prove that $ab + 1$ can be written in the form $ax + by$ for some *positive* integers x and y.
4. The *Fibonacci sequence* $1, 1, 2, 3, 5, 8, 13, \ldots$ is defined by the rule $f_1 = f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 3$.
   (a) Use Euclid's algorithm to determine $\gcd(f_n, f_{n-1})$ for all $n > 1$.
   (b) Define the *height* of an integer $a \geq 2$ to be the greatest integer n such that Euclid's algorithm requires n steps to determine $\gcd(a, b)$, where b can be any integer with $0 < b < a$. Show that the height of f_n is $\geq n - 2$ for all n.
   (c) Compute the height of all positive integers $a \leq 8$.