Math 250 – Number Theory – Homework 3

Due: Friday February 24th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Are the following statements true or false? In each case, give a proof or a counterexample.
   (a) If $\gcd(a, p^2) = p$ then $\gcd(a^2, p^2) = p^2$.
   (b) If $\gcd(a, p^2) = p$ and $\gcd(b, p^2) = p^2$ then $\gcd(ab, p^4) = p^3$.
   (c) If $\gcd(a, p^2) = p$ and $\gcd(b, p^2) = p$ then $\gcd(ab, p^4) = p^2$.
   (d) If $\gcd(a, p^2) = p$ then $\gcd(a + p, p^2) = p$.
2. (a) Consider k positive integers $a_1, a_2, \ldots, a_k$, such that $\gcd(a_i, a_j) = 1$ for all $i \neq j$. Prove that the product $a_1 \cdots a_k$ can be written in the form n^m for some integer m if and only if each a_i can be written in the form n_i^m for some integer n_i.
   (b) Does this result still hold true if some pair of numbers a_i, a_j for $i \neq j$ is not coprime? Why or why not?
   (c) If m and n are positive integers, under what circumstances is $m^{1/n}$ a rational number?
3. (a) Prove carefully that there are infinitely many primes of the form $3k + 2$, for $k \in \mathbb{N}$.
   (b) Does your proof also work for primes of the form $3k + 1$? If not, why not?
4. Prove that, for all positive integers k, you can find a sequence of k consecutive integers, none of which are prime.
5. Find a positive integer a with the property that $n^4 + a$ is never prime, for any $n \in \mathbb{Z}$.