Math 250 – Number Theory – Homework 4

Due: Friday March 10th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. (a) Prove that there is no polynomial $f(x)$ of degree at least 1 so that $f(x)$ is prime for all integers x.
   (b) Find a polynomial $f(x)$ of degree at least 1 so that $f(x)$ is prime when $x = 0, 1, 2$ and 3.
2. For each of the following congruences $ax \equiv b \mod n$, determine whether a solution exists. If a solution does exist, find all solutions in $\mathbb{Z}$, and the congruence classes they fall into in $\mathbb{Z}/n$.
   (a) $3x \equiv 5 \mod 7$.
   (b) $12x \equiv 15 \mod 22$.
   (c) $19x \equiv 42 \mod 50$.
   (d) $18x \equiv 42 \mod 50$.
3. (a) Prove that an integer a is divisible by 9 if and only if the sum of its decimal digits is divisible by 9.
   (b) Prove that an integer a is divisible by 11 if and only if the alternating sum of its decimal digits is divisible by 11. Recall that the "alternating sum" of a sequence $d_1, d_2, \ldots, d_k$ is
   $$d_1 - d_2 + d_3 - \cdots + (-1)^{k+1} d_k.$$
4. Let n be an integer with the property that $[-3]_n \neq [a]_n^2$ for any integer a.
   (a) Prove that the congruence $4(x^2 + x + 1) \equiv 0 \mod n$ has no solutions (Hint: complete the square.)
   (b) Hence show that the only solutions to the congruence $x^3 - 1 \equiv 0 \mod n$ are those where $[x] = [1]$.