Math 250 – Number Theory – Homework 7

Due: Friday April 14th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Compute the following residues. In each case, express your answer using the least positive residue.
   (a) $4^{31} \mod 7$.
   (b) $2^{1000} \mod 13$.
   (c) $2^{20} + 3^{30} + 4^{40} + 5^{50} + 6^{60} \mod 7$.
2. (a) Use Wilson's theorem to find $67! \mod 71$. Give your answer using the least positive residue.
   (b) Write
   $$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{22} + \frac{1}{23} = \frac{x}{23!}$$
   where x is some integer. Find the least positive residue of x modulo 13.
3. (a) Use Korselt's criterion to show that no number of the form pq, where p and q are primes, is ever a Carmichael number (Hint: show that if $p-1|pq-1$ then $p-1|q-1$).
   (b) Use Korselt's criterion to show that there are no Carmichael numbers less than 561.
   (c) Show that 10585 is a Carmichael number.
4. (a) Prove that for all natural numbers $n > 1$ we have
   $$\phi(n) = n\prod_{p|n}\left(1 - \frac{1}{p}\right)$$
   where the product is over all primes p dividing n.
   (b) Compute $\phi(168300)$.