Math 271 – Linear Algebra – Homework 1

Due: Friday February 7th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Let $\boldsymbol{u} = (1, 3, 2), \boldsymbol{v} = (-2, 3, 4), \boldsymbol{w} = (-3, 0, 3)$ in $\mathbb{R}^3$.

   (a) Compute $4\boldsymbol{u} - \boldsymbol{v}$.

   (b) Compute $-\boldsymbol{u} + \boldsymbol{v} + 3\boldsymbol{w}$,

2. For each of the following examples, either check that the axioms defining a vector space are satisfied, or explain why one of the axioms is *not* satisfied.

   (a) (See Example 1.1.3 in the textbook.) The set $P_n(\mathbb{R})$ of all real polynomials of degree n or less

   $$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \text{ for } a_i \in \mathbb{R},$$

   with addition and scalar multiplication given by the usual addition and multiplication of polynomials.

   (b) The set $P_{=n}(\mathbb{R})$ of all real polynomials of degree *exactly* n (so in the notation above, a_n is not zero), again with addition and scalar multiplication given by the usual addition and multiplication of polynomials.

   (c) The set $F(\mathbb{R})$ of all functions $f\colon \mathbb{R} \to \mathbb{R}$, with the usual scalar multiplication of functions

   $$(c \cdot f)(x) = c \cdot (f(x)),$$

   and the usual addition of functions

   $$(f + g)(x) = f(x) + g(x).$$

   (d) The set $F(\mathbb{R})$ of all functions $f\colon \mathbb{R} \to \mathbb{R}$, with the usual scalar multiplication of functions as above, but with vector addition given by *composition* of functions,

   $$(f \oplus g)(x) = f(g(x)).$$

3. Let V denote the set of all pairs (x, y) of real numbers, just like the vector space $\mathbb{R}^2$. Suppose we tried to define addition and scalar multiplication on V using the following rules:

   $$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, 0)$$
   $$c \odot (x, y) = (cx, 0).$$

   Do the operations $\oplus$ and $\odot$ make V into a vector space? Why or why not?

4. Prove that if V is any vector space, the additive identity $\mathbf{0}$ is *unique*. In other words, prove that if $\mathbf{0}$ and $\mathbf{0}'$ are two elements both satisfying the condition

   $$\mathbf{0} + \boldsymbol{v} = \boldsymbol{v} = \boldsymbol{v} + \mathbf{0} \text{ for all } \boldsymbol{v} \in V,$$

   then you must have $\mathbf{0} = \mathbf{0}'$.

5. Prove that in any vector space V

   (a) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$ then $\boldsymbol{v} = \boldsymbol{w}$.

   (b) If $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\lambda, \mu \in \mathbb{R}$ then

$$(\lambda + \mu)(\boldsymbol{u} + \boldsymbol{v}) = \lambda\boldsymbol{u} + \lambda\boldsymbol{v} + \mu\boldsymbol{u} + \mu\boldsymbol{v}.$$