Math 271 – Linear Algebra – Homework 3

Due: Friday February 21st

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Find all solutions to the following systems of linear equations by reducing to echelon form:

   (a)

   $$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 &= 4 \\ x_2 + x_3 &= 1 \end{aligned}$$

   (b)

   $$\begin{aligned} 4x_1 + 2x_2 - x_3 + x_4 &= -1 \\ x_1 + x_2 + 2x_4 &= 2 \\ 6x_1 + 4x_2 - x_3 + 5x_4 &= 3 \end{aligned}$$

   (c)

   $$\begin{aligned} x_1 + 2x_3 + x_5 &= 0 \\ -3x_1 + x_4 &= 1 \end{aligned}$$

2. Let $n \geq 4$. Let $U \subseteq \mathbb{R}^n$ be the subset consisting of all vectors satisfying

   $$x_i + x_{i+1} + x_{i+2} + x_{i+3} = 0,$$

   for all $i = 1, 2, 3, \ldots, n-3$.

   (a) Prove that U is a subspace of $\mathbb{R}^n$.
   (b) Find a basis for U in the case where $n = 8$.

3. Let S_1 and S_2 be subsets of a vector space V, and suppose that S_1 is linearly dependent and S_2 is linearly dependent.

   (a) Is $S_1 \cup S_2$ necessarily linearly dependent? Why or why not?
   (b) Is $S_1 \cap S_2$ necessarily linearly dependent? Why or why not?
   (c) Define the *symmetric difference* $S_1 \Delta S_2$ of S_1 and S_2 to be the set of all elements in *either* S_1 *or* S_2 but not both. Is $S_1 \Delta S_2$ necessarily linearly dependent? Why or why not?

4. Let $U \subseteq V$ be a subspace of a vector space V. We say that another subspace $U' \subseteq V$ is a *complementary* subspace for U if $U \cap U' = \{0\}$ and $U + U' = V$.

(a) Find a complementary subspace for the subspace $\mathrm{Span}((1,1,0,0),(0,0,1,1))$ of $\mathbb{R}^4$.

(b) Suppose that U_1 and U_2 are subspaces of a finite-dimensional vector space V, and U_1 and U_2 have *different* dimensions. Prove that there does not exist a subspace U' which is complementary to U_1 *and* complementary to U_2 (*Hint*: prove that if U is a subspace of V and U' is a complementary subspace to U then $\dim(U) + \dim(U') = \dim(V)$).

(c) If U_1 and U_2 are $n-1$-dimensional subspaces of a vector space V of dimension n, prove that there *does* exist a subspace U' of V that is complementary to U_1 and complementary to U_2.

(d) $(*)$ (*Optional.*) More generally prove that if U_1 and U_2 are any subspaces of a vector space n with the same dimension, then there does exist a subspace U' of V that is complementary to U_1 and complementary to U_2.

5. Define two subspaces of $\mathbb{R}^5$ in the following way:

$$V = \{x \in \mathbb{R}^5 \colon x_1 + x_3 + x_4 = 0, 2x_1 + 2x_2 + x_5 = 0\}, \text{ and } W = \{x \in \mathbb{R}^5 \colon x_1 + x_5 = 0, x_2 = x_3 = x_4\}.$$

(a) Find a basis for $V \cap W$.

(b) Find bases for V and for W that include your basis for $V \cap W$ as a subset.

(c) Hence find a basis for $V + W$.

(d) Show that

$$V + W = \{x \in \mathbb{R}^5 \colon x_1 + 2x_2 + x_5 = x_3 + x_4\}.$$