Math 271 – Linear Algebra – Homework 4

Due: Friday March 7th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Recall from homework 1 that $P_2(\mathbb{R})$ denotes the vector space of all polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree at most 2. Give an example of a linear map from $P_2(\mathbb{R})$ to $\mathbb{R}^3$ with rank equal to
   (a) 3
   (b) 2
   (c) 1
   (d) 0.

2. Show that the subset
$$U = \{p \in P_5(\mathbb{R})\colon p(1) = p(-1) = 0\}$$
is a subspace of $P_5(\mathbb{R})$ and find a basis for U.

3. Give bases for $\operatorname{Ker}(F)$ and $\operatorname{Im}(F)$ where $F\colon \mathbb{R}^3 \to \mathbb{R}^4$ has the following matrix with respect to the standard basis:
$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \\ 4 & 4 & 5 \\ 0 & -4 & -3 \end{pmatrix}.$$

4. If V and W are vector spaces, let $L(V, W)$ denote the set of all linear maps from V to W. We can add linear maps using the rule
$$(F + G)(v) = F(v) + G(v),$$
and we can multiply a linear map F by a scalar c using the rule
$$(cF)(v) = c \cdot F(v).$$
   (a) Show that these rules for addition and scalar multiplication make $L(V, W)$ into a vector space.
   (b) A commonly occurring special case is the case where $W = \mathbb{R}$, i.e. the space $L(V, \mathbb{R})$. We denote $L(V, \mathbb{R})$ by V^*, and call it the *dual space* of V.
   If the dimension of V is n, show that the dimension of V^* is also n (Hint: if $\{v_1, \ldots, v_n\}$ is a basis for V then every element of V can be written uniquely as a linear combination of elements of V. Use this fact to cook up a basis for V^* with the same size).
   (c) $(*)$ (*Optional.*) Determine the dimension of $L(\mathbb{R}^n, \mathbb{R}^m)$.