Math 271 – Linear Algebra – Homework 5

Due: Friday March 14th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Let $F\colon V \to W$ be a linear map between finite-dimensional vector spaces.
   (a) Show that if F is injective then $\dim(V) \leq \dim(W)$. Show that the converse is not true by finding a counterexample.
   (b) Show that if F is surjective then $\dim(V) \geq \dim(W)$. Show that the converse is not true by finding a counterexample.
2. Let $V = P_3(\mathbb{R})$ and let $W = P_4(\mathbb{R})$. Let $D\colon W \to V$ be the derivative map, $D(p) = p'$. Let $I\colon V \to W$ be the integral map $I(p) = \int_0^x p(t)\mathrm{d}t$. Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2, x^3, x^4\}$ be bases for V and W.
   (a) Compute the matrices $[D]_\beta^\alpha$ and $[I]_\alpha^\beta$.
   (b) Compute the matrices associated to the composite maps $[D \circ I]_\alpha^\alpha$ and $[I \circ D]_\beta^\beta$. What theorem of calculus is reflected in your answer to this part?
3. Suppose that $F\colon V \to W$ is a linear map, and suppose that $G\colon W \to V$ is an inverse for F, meaning $F \circ G(w) = w$ for all $w \in W$ and $G \circ F(v) = v$ for all $v \in V$. Prove that G is also linear.
4. Let V be a finite-dimensional vector space, and let $F\colon V \to V$ be a linear map. Write F^k for the map $F \circ F \circ \cdots \circ F$ obtained by composing F with itself k times. Let $r_k = \operatorname{rank}(F^k)$.
   (a) Show that $r_k \geq r_{k+1}$ for all positive integers k.
   (b) Give an example of a linear map $F\colon \mathbb{R}^4 \to \mathbb{R}^4$ where $r_1 > r_2 > r_3$.
   (c) $(*)$ *Optional*: Suppose $r_3 = 0$ and $r_2 > 0$. Show that $r_1 - r_2 \geq r_2$. Hint: you can always find a basis for V

   $$\{u_1, \ldots, u_k, v_1, \ldots, v_\ell, w_1, \ldots, w_m\}$$

   with the property that the vectors u_i are in the kernel of F, the vectors v_i are in the kernel of F^2 but not the kernel of F, and the vectors w_i are not in the kernel of F^2. Think about what the map F does to these basis vectors.
   (d) $(*)$ *Optional*: If $V = \mathbb{R}^4$, $r_3 = 0$ and $r_2 > 0$ show that we must have $r_2 = 1$ and $r_1 = 2$ or 3.