Math 271 – Linear Algebra – Homework 6

Due: March 28th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. (a) Let
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
be the matrix representing a linear map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ using the standard basis $\alpha = \{(1,0),(0,1)\}$ for $\mathbb{R}^2$. Compute $[F]_{\alpha'}^{\alpha'}$ where $\alpha' = \{(1,1),(1,-1)\}$.

   (b) Let
$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 0 & 0 \end{pmatrix}$$
be the matrix representing a linear map $G\colon \mathbb{R}^3 \to \mathbb{R}^3$ using the standard basis $\alpha = \{(1,0,0),(0,1,0),(0,0,1)\}$ for $\mathbb{R}^2$. Compute $[G]_{\alpha'}^{\alpha'}$ where $\alpha' = \{(1,0,1),(0,1,1),(1,1,0)\}$.

2. In this exercise, we will prove that isomorphism defines an *equivalence relation* on the set of all vector spaces. See Appendix A in the textbook for more on this idea. Recall that we say "V is isomorphic to W", written $V \cong W$, if there exists an isomorphism $F\colon V \to W$.

   (a) ("Reflexivity":) Prove that every vector space V is isomorphic to itself.

   (b) ("Symmetry":) Prove that if V is isomorphic to W then W is isomorphic to V.

   (c) ("Transitivity":) Prove that if U is isomorphic to V and V is isomorphic to W, then U is isomorphic to W.

3. Define a linear map $F\colon \mathbb{R}^4 \to \mathbb{R}^4$ by
$$F(x_1,x_2,x_3,x_4) = (-x_2, x_1 - x_2, x_3, -x_4).$$

   (a) Find a positive integer k with the property that $F^k = \mathrm{id}_{\mathbb{R}^4}$.

   (b) Prove that F is an isomorphism by determining the inverse map F^{-1}.

4. (a) Let $F\colon V \to W$ be a linear maps between finite-dimensional vector spaces. Let $Q\colon V \to V$ and $P\colon W \to W$ be isomorphisms. Show that $\mathrm{nullity}(F) = \mathrm{nullity}(P \circ F \circ Q)$ and $\mathrm{rank}(F) = \mathrm{rank}(P \circ F \circ Q)$.

   (b) Two linear maps F and $G\colon V \to V$ are called *conjugate* or *similar* if $G = Q^{-1} \circ F \circ Q$ for some isomorphism $Q\colon V \to V$. Show that if F, G are conjugate then F is an isomorphism if and only if G is an isomorphism.

   (c) In the setting of part (b), if F, G are conjugate isomorphisms, what is the relationship between F^{-1} and G^{-1}?

5. $(*)$ *Optional*: Let $L(V,W)$ be the vector space of linear maps $V \to W$ (as in homework 4, exercise 4). Let $\mathrm{Mat}_{m\times n}(\mathbb{R})$ be the vector space of $m \times n$ matrices, where $n = \dim(V)$ and $m = \dim(W)$. Find an isomorphism
$$\Phi\colon L(V,W) \to \mathrm{Mat}_{m\times n}(\mathbb{R}),$$
and hence compute the dimension of $L(V,W)$.