Math 271 – Linear Algebra – Homework 7

Due: Friday April 11th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Compute
$$\det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

2. Fix an $n \times n$ matrix A. Like we did in class, define a function
$$D\colon \mathrm{Mat}_{n\times n} \to \mathbb{R}$$
by the formula $D(B) = \det(AB)/\det(A)$.
   (a) Show that the function D is alternating in the columns of B.
   (b) Show that the function D is multilinear in the columns of B.
   (c) Hence deduce that for any two $n \times n$ matrices A and B, we have $\det(AB) = \det(A)\det(B)$.

3. *The Vandermonde determinant*. Consider the $n \times n$ matrix
$$V = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$
for $x_1, \ldots, x_n$ in $\mathbb{R}$. In this exercise we will prove
$$\det(V) = \prod_{1\le i<j\le n} (x_j - x_i)$$
where the product is over all pairs of integers (i, j) where i and j are between 1 and n, and $i < j$.
   (a) The determinant $\det(V)$ is a polynomial function of the variables $x_1, \ldots x_n$. What is its degree? Recall that the degree of a polynomial is the highest total power that appears, so for instance the polynomial $1 + x_1 + x_1^2x_2^2x_3^2$ has degree 6.
   (b) Prove that $x_j - x_i$ is a *factor* of $\det(V)$ for all $i < j$ (Hint: show that if $x_j - x_i = 0$ then $\det(V) = 0$ using the alternating property.)

(c) Hence show that

$$\det(V) = \prod_{1\le i<j\le n} (x_j - x_i).$$

4. (a) Show that if A is an n by n matrix and $c \in \mathbb{R}$ then

$$\det(cA) = c^n \det(A),$$

where by cA I mean the matrix obtained by multiplying all elements of the matrix A by c.

(b) Suppose that n is *odd*, and suppose that A is an n by n *skew-symmetric* matrix. That means that

$$A^T = -A.$$

Show that A is not invertible (Hint, you may use the fact that $\det(A^T) = \det(A)$).

(c) Give a counterexample to show that skew-symmetric n by n invertible matrices *do* exist if n is even (giving an example for the case $n = 2$ is enough).