Math 271 – Linear Algebra – Homework 8

Due: Friday April 18th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. Compute all the eigenvalues λ, together with bases for the associated eigenspaces E_λ, for the following matrices.

   (a)
$$\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$$

   (b)
$$\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

   (c)
$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 1 & 7 & 2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

2. Find necessary and sufficient conditions on real numbers a, b, c for the matrix
$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$
   to be diagonalizable.

3. Recall that if M is an $n \times n$ matrix, the *trace* of M, $\mathrm{Tr}(M)$, is the sum of the diagonal entries of M.

   (a) Show that a 2×2 matrix M has two distinct real eigenvalues if and only if
$$4\det(M) < (\mathrm{Tr}(M))^2.$$

   (b) Show that M has a single real eigenvalue if and only if
$$4\det(M) = (\mathrm{Tr}(M))^2.$$

   (c) What can you say about the eigenvalues of M if
$$4\det(M) > (\mathrm{Tr}(M))^2?$$

4. We proved in class that if P is any invertible matrix, $\mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(A)$

   (a) Use this to show that if M, N are any two matrices and N is invertible, then $\mathrm{Tr}(MN) = \mathrm{Tr}(NM)$.

   (b) Use this fact to show that there do *not* exist two invertible matrices M, N with the property that $MN - NM = I$.

   (c) $(*)$ *Optional*: Show that parts a) and b) still hold even if M and N are not invertible.

5. (a) If M is a 3×3 matrix, show that M has at least one real eigenvalue.

   (b) How can this result be generalized? In other words, for which values of n can you guarantee that an $n \times n$ matrix will have a real eigenvalue?