Math 271 – Linear Algebra – Homework 9

Due: Friday April 25th

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

1. For each of the following vector spaces V and functions $\langle , \rangle \colon V \times V \to \mathbb{R}$, either prove that $\langle , \rangle$ is an inner product or explain why it fails to satisfy the axioms of an inner product.

   (a) Let $V = \mathbb{R}^3$. For $\boldsymbol{x}, \boldsymbol{y} \in V$ define

   $$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_1y_2 + x_2y_1 + x_1y_3 + x_3y_1 + x_2y_3 + x_3y_2.$$

   (b) Let $V = \mathrm{Mat}_{n\times n}(\mathbb{R})$. For $A, B \in V$ define

   $$\langle A, B \rangle = \mathrm{Tr}(AB).$$

   (c) Let $V = \mathrm{Mat}_{n\times n}(\mathbb{R})$. For $A, B \in V$ define

   $$\langle A, B \rangle = \mathrm{Tr}(AB^T).$$

2. Find an orthogonal basis for the subspace $W \subseteq \mathbb{R}^5$ defined by

   $$W = \{\boldsymbol{x} \in \mathbb{R}^5 \colon x_1 + 2x_2 - x_3 + x_5 = 0, 3x_1 + x_3 + 3x_4 = 0\}.$$

   Don't be deterred if the numbers you get are not nice!

3. Let V be a vector space equipped with an inner product $\langle , \rangle$, and let $W \subseteq V$ be a subspace. Recall that the *orthogonal projection* onto W is defined to be the linear map $\pi_W : V \to V$ given by the formula

   $$\pi_W(v) = \sum_{i=1}^{m} \langle w_i, v \rangle w_i,$$

   where $\{w_1, \dots, w_m\}$ is an orthonormal basis for W.

   (a) Let $\alpha = \{w_1, \dots, w_m, v_{m+1}, \dots, v_n\}$ be an orthonormal basis for V extending the given orthonormal basis for W. Describe the matrix $M = [\pi_W]_\alpha^\alpha$.

   (b) Prove that $(\pi_W)^2 = \pi_W$.

4. (*Legendre polynomials*): Define an inner product on the vector space $V = P_n(\mathbb{R})$ by

   $$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\mathrm{d}x.$$

   Let $s_k(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left((1-x^2)^k\right)$, for $k = 0, \dots, n$.

(a) $(*)$ *Optional*: Use integration by parts to show that $\{s_0(x), \ldots, s_n(x)\}$ form an orthogonal basis for $P_n(\mathbb{R})$.

(b) Show that the polynomials s_k for $k = 0, 1, 2, 3$ are eigenvectors of the linear map $A\colon V \to V$ given by

$$A(f)(x) = (1 - x^2)f''(x) - 2xf'(x).$$

Compute their eigenvalues. ($(*)$ *Optional*: check it for all k.)

(c) Let $n = 3$. Apply the Gram–Schmidt process to the basis $\{1, x, x^2, x^3\}$. Comment on the relationship to the set s_k.