Math 350 – Groups, Rings and Fields – Homework 1

Due: Friday February 9th at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. (a) Prove that if A_1, A_2 are subsets of a set B then
$$(B \setminus A_1) \cap (B \setminus A_2) = B \setminus (A_1 \cup A_2).$$
   (b) For any $n \geq 1$, prove by induction that if $A_1, \ldots, A_n$ are subsets of a set B then
$$(B \setminus A_1) \cap \cdots \cap (B \setminus A_n) = B \setminus (A_1 \cup \cdots \cup A_n).$$

2. Recall that the *power set* $\mathbb{P}(S)$ of a set S is the set of all subsets of S. Prove that if S has n elements then $\mathbb{P}(S)$ has 2^n elements.

3. Write $\mathbb{Q}^\times$ for the set $\mathbb{Q} \setminus \{0\}$ of non-zero rational numbers. Prove that $\mathbb{Q}^\times$ is not a group under the operation $\div$. Which axioms in the definition of a group fail to hold? Think carefully about the identity axiom in particular.

4. The *symmetric difference* of two subsets U and V of a set S is defined to be
$$U \Delta V = (U \cup V) \setminus (U \cap V).$$
   Prove that the operation Δ makes $\mathbb{P}(S)$ into a group.

B

5. Let $*$ be an associative binary operation on a set S, and define the *center* $Z(S)$ of S to be
$$Z(S) = \{z \in S \colon z * x = x * z \text{ for all } x \in S\},$$
   the set of elements that commute with everything in S.
   (a) Prove that $Z(S)$ is closed under the operation $*$.
   (b) Suppose that $(S, *)$ is a group. Prove that $(Z(S), *)$ is an abelian group.

6. (a) Let S be a set with n elements. Consider the set Bin_S consisting of all binary operations on S. How many elements does Bin_S have? (Hint: more generally, what is the size of the set of functions from a set T to a set S in terms of the numbers of elements of S and T?)
   (b) Let ComBim_S denote the subset of Bin_S consisting of all *commutative* binary operations on S. How many elements does ComBin_S have?

7. $(*)$ *Optional:* Let $*$ be an associative binary operation on the set S. Let $x_1, \ldots, x_n$ be any ordered sequence of elements of S. Prove that the n-fold product

$$x_1 * x_2 * \cdots * x_n$$

does not depend on where we put the brackets (so the case where $n = 3$ is the usual associativity law).