Math 350 – Groups, Rings and Fields – Homework 10 (§19–20)

Due: Friday May 3rd at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. Determine which of the following polynomials in $\mathbb{Q}[x]$ are irreducible.
   (a) $x^3 + x + 36$
   (b) $2x^4 + 3x^2 + 15x + 6$
   (c) $x^4 + x^3 + x^2 + x + 1 = (x^5 - 1)/(x - 1)$ (Hint: compare to Example 3 on p200 of Saracino)
   (d) $x^6 + x^2 - 2$.

2. For $f, g \in \mathbb{Z}[x]$ both non-zero, prove that

$$\mathrm{content}(f)\mathrm{content}(g) = \mathrm{content}(fg).$$

3. (a) Let $f \in \mathbb{R}[x]$, and let $c = a + ib \in \mathbb{C}$ be a root of f in $\mathbb{C}$. Prove that $\bar{c} = a - ib$ is also a root of f.
   (b) Prove that f may be factored in $\mathbb{R}[x]$ into a product of irreducible polynomials each of which has degree 1 or 2. You may use the *fundamental theorem of algebra*, which says that every polynomial in $\mathbb{C}[x]$ may be factored into a product of irreducible polynomials all of which have degree 1.

B

4. Let a be a natural number. Prove that $x^4 + a \in \mathbb{Q}[x]$ is reducible if and only if $a = 4b^4$ for some integer b.

5. (a) Let $f \in \mathbb{Z}/p[x]$ be irreducible of degree n and let $K = \mathbb{Z}/p[x]/(f)$. Prove that K is a field with p^n elements.
   (b) Use part a to construct a field with 9 elements and a field with 125 elements.

6. Let F be a finite field, prove that $|F| = p^n$ for some prime number p and natural number n (Hint: let $K \subseteq F$ be the smallest subfield of F. Prove that $|K|$ is prime, and consider F as a vector space over K).

7. $(*)$ *Optional*: We will prove that there is a field of size p^n for every prime p and natural number n.
   (a) Let $f(x) = x^{p^n} - x \in \mathbb{Z}/p[x]$. Let K be a field containing $\mathbb{Z}/p$ in which f has p^n roots $c_1, \ldots, c_{p^n}$, by Kronecker's theorem. Prove that all these roots are distinct (use Exercise 5b on Homework 9).
   (b) Prove that $F = \{c_1, \ldots, c_{p^n}\}$ is a subfield of K containing $\mathbb{Z}/p$.