Math 350 – Groups, Rings and Fields – Homework 2 (Section 2–3)

Due: Friday February 16th at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. Let $X = \{1, 2, 3, \ldots, 11, 12\}$, and consider the group $(\mathbb{P}(X), \Delta)$. If $A = \{1, 3, 5, 7, 8\}$ and $B = \{2, 3, 6\}$, solve the equation $A \Delta U = B$ for $U \in \mathbb{P}(X)$.

2. Prove that in any group $(G, *)$ the left and right *cancellation laws* hold. That is, prove that

   $$\text{if } x * y = x * z \text{ then } y = z,$$
   $$\text{and if } y * x = z * x \text{ then } y = z,$$

   for all x, y and z in G.

3. Let $G = \{x \in \mathbb{R} \colon x \neq -1\}$. Define a binary operation on G by

   $$x * y = x + y + xy.$$

   (a) Prove that $(G, *)$ is a group.

   (b) Find the inverse of the elements 2 and 3 in G.

   (c) Solve the equation $2 * x * 5 = 6$ for x in G.

B

4. Let $(G, *)$ be a group. Prove that G is abelian if and only if $(g * h)^2 = g^2 * h^2$ for all $g, h \in G$, where $g^2 = g * g, h^2 = h * h$.

5. (a) Suppose that $(G, *)$ is a group where every element $g \in G$ has the property that $g = g^{-1}$. Prove that G is abelian.

   (b) Suppose that G is a finite group and $|G|$ is even. Prove that there exists some element of G other than e such that $g = g^{-1}$ (Hint: prove the contrapositive, i.e. suppose that no elements satisfy $g = g^{-1}$ and deduce that $|G|$ cannot be even).

6. $(*)$ *Optional*: Let G be a non-empty set equipped with an associative binary operation $*$. Suppose that the left and right cancellation laws from problem 2 hold in $(G, *)$. Then:

   (a) If G is finite, prove that $(G, *)$ is a group. (Hint: start by trying to identify a unit. Choose an element $x \in G$, and think about the set of all powers x^n of x where $x^n = x * x * \cdots * x$ with x appearing n times.)

   (b) If G is allowed to be infinite, find a counterexample to show that $(G, *)$ does not need to be a group.