Math 350 – Groups, Rings and Fields – Homework 3 (Section 4–5)

Due: Friday February 23rd at 5pm

Please explain your answers carefully using full sentences, not only symbols. You may use the textbook and your notes, and you're welcome to discuss the problems with one another or with me. However, your final answers should be written on your own and in your own words.

At the top of the first page, please list any classmates you collaborated with while working on these exercises (so that we know to expect similar solutions).

A

1. Prove that if G is any group and $g \in G$ is any element then
$$g^{ab} = (g^a)^b$$
for all integers a and b.
2. Prove that
   (a) If $G = \langle x \rangle$, prove that $G = \langle x^{-1} \rangle$ also.
   (b) If $G = \langle x \rangle$ is an *infinite* cyclic group, prove that if $G = \langle y \rangle$ then $y = x$ or $y = x^{-1}$.
3. (a) List all the subgroups of the group $\mathbb{Z}/36$. Explain why your list is complete.
   (b) List all the subgroups of the Klein four group V. Explain why your list is complete.

B

4. Let $G = \mathrm{GL}(n, \mathbb{Q})$ denote the group of n by n matrices with rational coefficients.
   (a) Is the group G cyclic for any positive integer n?
   (b) For $n = 2$ describe two different cyclic subgroups of G.
5. Let G be a finite subgroup of the group $(\mathbb{C}^\times, \times)$ of non-zero complex numbers. Prove that G is cyclic.
6. If G is a group and $n \in \mathbb{Z}_{>0}$, define
$$G_n = \{g^n \colon g \in G\}.$$
Prove that G_n is a subgroup of G for all n if G is abelian. Give a counter-example to show that this does not need to be true if G is non-abelian.